On WOWA Rank Reversal

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Abstract. The problem of aggregating multiple criteria to form an overall measure is of considerable importance in many disciplines. The ordered weighted averaging (OWA) aggregation, introduced by Yager, uses weights assigned to the ordered values rather than to the specific criteria. This allows one to model various aggregated preferences, preserving simultaneously the impartiality (neutrality) with respect to the individual criteria. However, importance weighted averaging is a central task in multicriteria decision problems of many kinds. It can be achieved with the Weighted OWA (WOWA) aggregation, introduced by Torra, covering both the weighted means and the OWA averages as special cases. In this paper we analyze the monotonicity properties of the WOWA aggregation with respect to changes of importance weights. In particular, we demonstrate that a rank reversal phenomenon may occur in the sense that increasing the importance weight for a given criterion may enforce the opposite WOWA ranking than that imposed by the criterion values.

Keywords: OWA, WOWA, Multicriteria Optimization, Rank Reversal

1 Introduction

Consider a decision problem defined by m criteria. That means the decisions are characterized by m-dimensional outcome vectors $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$. In order to make the multicriteria model operational for the decision support process, one needs to assume some aggregation function $a : R^m \to R$. The aggregated value can then be optimized (maximized or minimized).

The most commonly used aggregation is based on the weighted mean where positive importance weights p_i $(i = 1, \ldots, m)$ are allocated to several criteria

$$A_{\mathbf{p}}(\eta) = \sum_{i=1}^{m} p_i \eta_i \tag{1}$$

The weights are typically normalized to the total 1 ($\sum_{i=1}^{m} p_i = 1$). However, the weighted mean allowing to define the importance of criteria does not allow to model the decision maker's preferences regarding the distribution of outcomes. The latter is crucial when aggregating (normalized) uniform achievement criteria like those used in the fuzzy optimization methodologies [19] as well as in the goal programming and the reference point approaches to multiple criteria decision support [7]. In stochastic problems uniform objectives may represent various possible values of the same (uncertain) outcome under several scenarios [6].

The preference weights can be effectively introduced with the so-called Ordered Weighted Averaging (OWA) aggregation function developed by Yager [16]. In the OWA aggregation the weights are assigned to the ordered values (i.e. to the smallest value, the second smallest and so on) rather than to the specific criteria. Since its introduction, the OWA aggregation has been successfully applied to many fields of decision making [19, 20].

The OWA operator allows to model various aggregation functions from the maximum through the arithmetic mean to the minimum. Thus, it enables modeling of various preferences from the optimistic to the pessimistic one. On the other hand, the OWA does not allow to allocate any importance weights to specific criteria. Actually, the weighted mean (1) cannot be expressed in terms of the OWA aggregations.

Importance weighted averaging is a central task in multicriteria decision problems of many kinds, such as selection, classification, object recognition, and information retrieval. Therefore, several attempts have been made to incorporate importance weighting into the OWA operator [18, 2]. Finally, Torra [13] has introduced the Weighted OWA (WOWA) aggregation defined by two weighting vectors: the preferential weights $\mathbf{w}$ and the importance weights $\mathbf{p}$. It covers both the weighted means (defined with $\mathbf{p}$) and the OWA averages (defined with $\mathbf{w}$) as special cases. Actually, the WOWA average is reduced to the weighted mean in the case of equal preference weights and it becomes the standard OWA average in the case of equal importance weights. Since its introduction, the WOWA operator has been successfully applied to many fields of decision making [15, 9, 10, 7, 8] including metadata aggregation problems [1, 5].

While considering the importance weighting of the criteria one may expect some monotonicity properties of the aggregation with respect to the (relative) increase of a given importance weight. The basic stability requirements with respect to a given importance weight can be formalized as two properties: rank stability and asymptotic monotonicity. We say that an aggregation satisfies the rank stability property if whenever the aggregation ranks two vectors consistently with the inequality on a given criterion it preserves this ranking for any positive increase of importance weight for the given criterion. If despite that for some importance weights the aggregation ranks two vectors consistently with the relation on a given criterion, a positive increase of importance weight for the given criterion may result in an opposite inequality, we say that the rank reversal phenomenon occurs. We say that an aggregation satisfies the asymptotic monotonicity property if for any importance weights independently from the relation between the aggregation values for two vectors, a sufficiently large increase of the importance weight for a given criterion enforces the aggregation ranking consistently with the inequality on the given criterion values. Both stability properties are satisfied by the weighted mean (1). We analyze how the WOWA aggregation models the importance weighting stability properties. Unfortunately, we are able to show a possible rank reversal phenomenon which may be considered a serious flaw of the WOWA importance weighting scheme. However, the WOWA aggregation fulfills the asymptotic monotonicity property.

The paper is organized as follows. In the next section we formally introduce the WOWA operator and recall some alternative computational formula based on the Lorenz curves. In Section 3 we show some examples of the rank reversal phenomenon for the WOWA aggregation. Next, in Section 4 we prove the asymptotic monotonicity showing required levels for sufficiently large increase of the importance weight for various special cases of the WOWA aggregation.

2 WOWA Aggregation

Let $\mathbf{w} = (w_1, \ldots, w_m)$ be a weighting vector of dimension m such that $w_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} w_i = 1$. The corresponding OWA aggregation of outcomes $\eta = (\eta_1, \ldots, \eta_m)$ can be mathematically formalized as follows [16]. Let $\langle\eta\rangle = (\eta_{\langle 1\rangle}, \eta_{\langle 2\rangle}, \ldots, \eta_{\langle m\rangle})$ denote the vector obtained from η by rearranging its components in the non-increasing order. That means $\eta_{\langle 1\rangle} \geq \eta_{\langle 2\rangle} \geq \cdots \geq \eta_{\langle m\rangle}$ and there exists a permutation τ of set $I = \{1, \ldots, m\}$ such that $\eta_{\langle i\rangle} = \eta_{\tau(i)}$ for $i = 1, \ldots, m$. Further, we apply the weighted sum aggregation to ordered outcome vectors $\langle\eta\rangle$, i.e. the OWA aggregation has the following form:

$$OWA_{\mathbf{w}}(\eta) = \sum_{i=1}^{m} w_i \eta_{\langle i\rangle} \tag{2}$$

Due to the strict monotonicity of the OWA aggregation with positive weighting vectors [4], the OWA optimization generates a Pareto optimal solution.

The OWA aggregation (2) allows to model various aggregation functions from the maximum ($w_1 = 1$, $w_i = 0$ for $i = 2, \ldots, m$) through the arithmetic mean ($w_i = 1/m$ for $i = 1, \ldots, m$) to the minimum ($w_m = 1$, $w_i = 0$ for $i = 1, \ldots, m-1$). When the differences among weights tend to infinity, the OWA aggregation approximates the leximax ranking of the ordered outcome vectors [17]. However, the weighted mean (1) cannot be expressed as an OWA aggregation. Actually, the OWA aggregations are symmetric (impartial, neutral) with respect to the individual criteria and it does not allow to represent any importance weights allocated to specific criteria.

Importance weighted averaging is a central task in multicriteria decision problems of many kinds and the ordered averaging model enables one to introduce importance weights to affect achievement importance by rescaling accordingly its measure within the distribution of achievements as defined in the so-called Weighted OWA (WOWA) aggregation [13]. Let $\mathbf{w} = (w_1, \ldots, w_m)$ be OWA weights and let $\mathbf{p} = (p_1, \ldots, p_m)$ be an additional importance weighting vector such that $p_i \geq 0$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{m} p_i = 1$. The corresponding Weighted

OWA aggregation of achievements $\eta = (\eta_1, \ldots, \eta_m)$ is defined as follows [13]:

$$WOWA_{\mathbf{w},\mathbf{p}}(\eta) = \sum_{i=1}^{m} v_i(\mathbf{p}, \eta)\eta_{\langle i \rangle} \tag{3}$$

where weights v_i are defined as

$$v_i(\mathbf{p}, \eta) = \varphi(\sum_{k \leq i} p_{\tau(k)}) - \varphi(\sum_{k<i} p_{\tau(k)}) \tag{4}$$

with φ a monotone increasing function that interpolates points $(\frac{i}{m}, \sum_{k \leq i} w_k)$ together with point (0.0) and τ representing the ordering permutation for η (i.e., $\eta_{\tau(i)} = \eta_{\langle i \rangle}$). Moreover, function φ is required to be a straight line when the point can be interpolated in this way, thus allowing the WOWA to cover the standard weighted mean with weights p_i as a special case of equal OWA weights ($w_i = 1/m$ for $i = 1, \ldots, m$). Indeed, the WOWA defined by (3)–(4) as OWA aggregation with modified preferential weights may be rewritten as the weighted mean with modified weights:

$$WOWA_{\mathbf{w},\mathbf{p}}(\eta) = \sum_{i=1}^{m} \pi_i(\mathbf{p}, \eta)\eta_i \tag{5}$$

where the weights π_i are defined as

$$\pi_i(\mathbf{p}, \eta) = \varphi(p_i + \sum_{k<\tau(i)} p_{\tau(k)}) - \varphi(\sum_{k<\tau(i)} p_{\tau(k)}). \tag{6}$$

Actually, the WOWA aggregation is a special case of the rank dependent utility [12] with a piecewise linear probability weighting function φ defined by the importance weights.

The WOWA may be expressed with a more direct formula where preferential (OWA) weights w_i are applied to the averages of the corresponding portions of ordered outcomes (quantile intervals) according to the distribution defined by importance weights p_i [9]. Note that one may alternatively compute the WOWA values by using rational importance weights to replicate the corresponding achievements and then calculate the OWA aggregations. This approach can be generalized to any real importance weights and the WOWA aggregation can be equivalently defined as follows [9]:

$$WOWA_{\mathbf{w},\mathbf{p}}(\eta) = \sum_{i=1}^{m} w_i m \int_{\frac{i-1}{m}}^{\frac{i}{m}} \overline{F}_{\eta}^{(-1)}(\xi)\; d\xi \tag{7}$$

where $\overline{F}_{\eta}^{(-1)}$ is the stepwise function $\overline{F}_{\eta}^{(-1)}(\xi) = \eta_{\langle k \rangle}$ for $\sum_{j<k} p_{\tau(j)} < \xi \leq \sum_{j \leq k} p_{\tau(j)}$, for $k = 1, \ldots, m$ with τ representing the ordering permutation for η (i.e., $\eta_{\tau(k)} = \eta_{\langle k \rangle}$). It can also be mathematically formalized as the quantile function defined as the left-continuous inverse of the decumulative distribution function, i.e., $\overline{F}_\eta^{(-1)}(\xi) = \sup\ \{z : \overline{F}_\eta(z) \geq \xi\}$ for $0 < \xi \leq 1$ with $\overline{F}_\eta(z) = \sum_{i=1}^n p_i\zeta_i(z)$ where $\zeta_i(z) = 1$ if $\eta_i \geq z$ and 0 otherwise.

Formula (7), defining the WOWA value by applying preferential weights w_i to importance weighted averages within quantile intervals, may be reformulated with the tail averages (Lorenz components):

$$WOWA_{\mathbf{w},\mathbf{p}}(\eta) = \sum_{k=1}^{m} \overline{w}_k m L(\eta, \mathbf{p}, \frac{k}{m}) \quad \text{where} \quad L(\eta, \mathbf{p}, \xi) = \int_0^\xi \overline{F}_\eta^{(-1)}(\zeta) d\zeta \quad (8)$$

and differential weights

$$\overline{w}_k = w_k - w_{k+1} \quad \text{for } k = 1, \dots, m-1 \quad \text{and} \quad \overline{w}_m = w_m \quad (9)$$

Note that the differential weights $\overline{w}_i$ are positive in the case of positive and strictly decreasing preferential (OWA) weights $w_1 > w_2 > \dots > w_m > 0$. Graphs of functions $L(\eta, \mathbf{p}, \xi)$ (with respect to ξ) take the form of concave piecewise linear curves, the so-called (upper) absolute Lorenz curves. Moreover, values of function $L(\eta, \mathbf{p}, \xi)$ for any $0 \leq \xi \leq 1$ can be given by linear programming (LP) optimization which enables the WOWA minimization to be implemented with a LP model [9], in the case of positive and decreasing preferential (OWA) weights $w_1 \geq w_2 \geq \dots \geq w_m > 0$.

Applying the WOWA aggregation to a multiple criteria optimization problem we get the WOWA optimization model. For any positive weights $\mathbf{w}$ and $\mathbf{p}$, the WOWA aggregation is strictly monotonic [7]. Therefore, the WOWA optimal solutions are then Pareto-optimal.

3 Rank Reversal

When considering the importance weighting of the attributes one may expect some monotonicity properties of the aggregation with respect to changes of the importance weights. Note that for any vector of importance weights $\mathbf{p}$ any positive increase of a given importance weight must be accompanied by decrease of some other weights. We will focus on weights changes represented by a positive increase of a given importance weight p_{i_o} with proportional decrease of other weights, i.e., we will consider a parameterized importance weight modification

$$\mathbf{p}(\varepsilon) = \frac{1}{1+\varepsilon}(\mathbf{p} + \varepsilon \mathbf{e}_{i_o}) \quad \text{with} \quad \varepsilon > 0 \quad (10)$$

where $\mathbf{e}_i$ denotes the ith unit vector. The basic stability requirements with respect to a given importance weight can be formalized as two properties: rank stability and asymptotic monotonicity.

Rank stability and rank reversal. Let η' and η'' be vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$. We say that an aggregation satisfies the rank stability property if whenever for any importance weights $\mathbf{p}$, the aggregation of η' is less or equal to that for η'', then this inequality remains valid for any positive increase of importance weight p_{i_o} with proportional decrease of other weights. If despite that for some importance weights $\mathbf{p}$ the aggregation of η' is less than that for η'', a positive increase of importance weight p_{i_o} with proportional decrease of other weights may result in opposite inequality, we say that the rank reversal phenomenon occurs.

Asymptotic monotonicity. Let η' and η'' be vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$. We say that an aggregation satisfies the asymptotic monotonicity property if for any importance weights $\mathbf{p}$ independently from the relation between the aggregation values of η' and η'', a sufficiently large increase of importance weight p_{i_o} with proportional decrease of other weights enforces the aggregation ranking consistently with inequality $\eta'_{i_o} < \eta''_{i_o}$.

One may notice that both the stability properties are satisfied by the weighted mean (1). Indeed, for any vectors η', η'' and importance weights $\mathbf{p}$, while increasing importance weight p_{i_o} with proportional decrease of other weights, following (10) one gets

$$\begin{aligned} A_{\mathbf{p}(\varepsilon)}(\eta') - A_{\mathbf{p}(\varepsilon)}(\eta'') &= \sum_{i=1}^{m} p_i(\varepsilon)(\eta'_i - \eta''_i) \\ &= \frac{1}{1+\varepsilon}\sum_{i=1}^{m} p_i(\eta'_i - \eta''_i) + \frac{\varepsilon}{1+\varepsilon}(\eta'_{i_o} - \eta''_{i_o}) \\ &= \frac{1}{1+\varepsilon}(A_{\mathbf{p}}(\eta') - A_{\mathbf{p}}(\eta'')) + \frac{\varepsilon}{1+\varepsilon}(\eta'_{i_o} - \eta''_{i_o}) \end{aligned} \quad (11)$$

This leads to the following statements.

Proposition 1 *Let η' and η'' be outcome vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$. If $A_{\mathbf{p}}(\eta') \leq A_{\mathbf{p}}(\eta'')$ for some importance weights $\mathbf{p}$, then any positive increase of importance weight p_{i_o} with proportional decrease of other weights, following (10), results in strict inequality on averages $A_{\mathbf{p}(\varepsilon)}(\eta') < A_{\mathbf{p}(\varepsilon)}(\eta'')$.*

Proposition 2 *Let η' and η'' be outcome vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$. For any vector of importance weights $\mathbf{p}$, a sufficiently large increase of importance weight p_{i_o} with proportional decrease of other weights, following (10) with*

$$\varepsilon > \frac{\max\{A_{\mathbf{p}}(\eta') - A_{\mathbf{p}}(\eta''), 0\}}{\eta''_{i_o} - \eta'_{i_o}}$$

results in strict inequality on averages $A_{\mathbf{p}(\varepsilon)}(\eta') < A_{\mathbf{p}(\varepsilon)}(\eta'')$.

Unfortunately, the WOWA aggregation does not guarantee the rank stability. We will show that the rank reversal phenomenon may occur for the WOWA aggregation even in a simple case of ordered vectors. Consider two vectors $\eta' = (1000, 102, 10)$ and $\eta'' = (1000, 100, 12)$. While introducing preferential weights $\mathbf{w} = (0.8, 0.1, 0.1)$ and assuming an equal importance of all the criteria, i.e. $\mathbf{p} = (1/3, 1/3, 1/3)$, one gets:

$$\begin{aligned} WOWA_{\mathbf{w},\mathbf{p}}(\eta') &= OWA_{\mathbf{w}}(\eta') = 0.8 \cdot 1000 + 0.1 \cdot 102 + 0.1 \cdot 10 = 811.2 \\ WOWA_{\mathbf{w},\mathbf{p}}(\eta'') &= OWA_{\mathbf{w}}(\eta'') = 0.8 \cdot 1000 + 0.1 \cdot 100 + 0.1 \cdot 12 = 811.2 \end{aligned}$$

Thus with equally important criteria $WOWA_{\mathbf{w},\mathbf{p}}(\eta') = WOWA_{\mathbf{w},\mathbf{p}}(\eta'')$ and according to the ordered aggregation both the vectors are equally good.

Suppose one wish to consider criterion η_3 as much more important, say 4 times more important than those related to the first or second criterion. For this purpose, importance weights $\bar{\mathbf{p}} = (1/6, 1/6, 2/3)$ are introduced. Note that $\bar{\mathbf{p}}$ may be understood as a result of increasing p_3 by 1 and renormalizing all weights, i.e., $\bar{\mathbf{p}} = \frac{1}{1+\varepsilon}(\mathbf{p} + \varepsilon\mathbf{e}_3)$ with $\varepsilon = 1$. Since $\eta'_3 < \eta''_3$, one may expect $WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta') < WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta'')$. However this is not the case, as we show now.

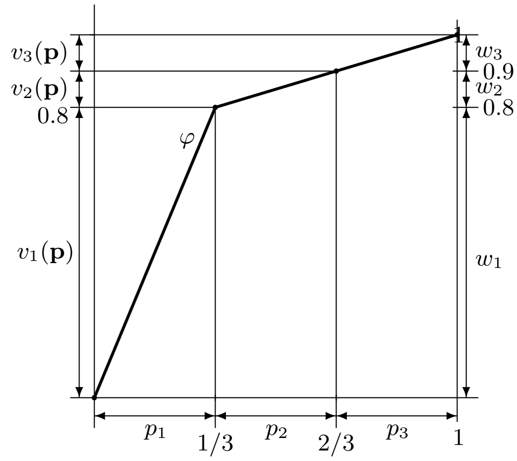

Fig. 1. Definition of function φ for $\mathbf{w} = (0.8, 0.1, 0.1)$ and WOWA weights v_i for equally important attributes $\mathbf{p} = (1/3, 1/3, 1/3)$ and vectors η with ordered coefficients $\eta_1 \geq \eta_2 \geq \eta_3$

To take into account the importance weights in the WOWA aggregation (3) we introduce the piecewise linear function φ (Fig. 1):

$$\varphi(\xi) = \begin{cases} 2.4\xi & \text{for } 0 \leq \xi \leq 1/3 \\ 0.8 + 0.3(\xi - 1/3) & \text{for } 1/3 < \xi \leq 2/3 \\ 0.9 + 0.3(\xi - 2/3) & \text{for } 2/3 < \xi \leq 1.0 \end{cases} \tag{12}$$

Actually, since vectors η' and η'' are both already ordered, the ordered weights v_i are identical for both of them $v_i(\mathbf{p}, \eta') = v_i(\mathbf{p}, \eta'') = v_i(\mathbf{p})$. In the case of equal importance weights $\mathbf{p} = (1/3, 1/3, 1/3)$, obviously, $v_i(\mathbf{p}) = w_i$ (as presented in Fig. 1). Calculating weights v_i according to formula (4) with function φ given by (12), as illustrated in Fig. 2, one gets $v_1(\bar{\mathbf{p}}) = \varphi(1/6) = 0.4$, $v_2(\bar{\mathbf{p}}) = \varphi(1/3) - \varphi(1/6) = 0.4$ and $v_3(\bar{\mathbf{p}}) = 1 - \varphi(1/3) = 0.2$. Hence,

$$\begin{aligned} WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta') &= 0.4 \cdot 1000 + 0.4 \cdot 102 + 0.2 \cdot 10 = 442.8 \\ WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta'') &= 0.4 \cdot 1000 + 0.4 \cdot 100 + 0.2 \cdot 12 = 442.4 \end{aligned}$$

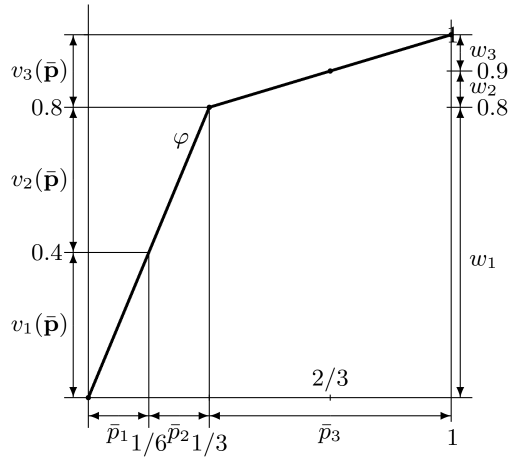

Fig. 2. Definition of WOWA weights v_i with $\mathbf{w} = (0.8, 0.1, 0.1)$ and $\bar{\mathbf{p}} = (1/6, 1/6, 2/3)$ for vectors η with ordered coefficients $\eta_1 \geq \eta_2 \geq \eta_3$

Compare with the same weights vector $\eta' = (1000, 102, 10)$ with $\eta''' = (1000, 100, 13)$. Assuming an equal importance of all criteria, i.e. $\mathbf{p} = (1/3, 1/3, 1/3)$, one gets:

$$WOWA_{\mathbf{w},\mathbf{p}}(\eta') = OWA_{\mathbf{w}}(\eta') = 0.8 \cdot 1000 + 0.1 \cdot 102 + 0.1 \cdot 10 = 811.2$$
$$WOWA_{\mathbf{w},\mathbf{p}}(\eta''') = OWA_{\mathbf{w}}(\eta''') = 0.8 \cdot 1000 + 0.1 \cdot 100 + 0.1 \cdot 13 = 811.3$$

Thus with equally important criteria $WOWA_{\mathbf{w},\mathbf{p}}(\eta')$ is a little bit smaller than $WOWA_{\mathbf{w},\mathbf{p}}(\eta''')$ similar to inequality on the third criterion $\eta'_3 < \eta'''_3$. Consider now criterion η_3 as 4 times more important than those related to the first or second criterion, i.e., importance weights $\bar{\mathbf{p}} = (1/6, 1/6, 2/3)$. Since, the vectors are already ordered, the corresponding ordered weights calculation remains the same as for the earlier comparison of vectors η' and η'' (see Fig. 2). Hence,

$$WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta''') = 0.4 \cdot 1000 + 0.4 \cdot 100 + 0.2 \cdot 13 = 442.6 < 442.8 = WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta')$$

Thus, despite that for equal importance weights the WOWA aggregation ranks vectors η' and η''' consistently with the inequality on the third criterion, increasing the importance weight for this criterion results in a rank reversal.

Since our examples are built on ordered vectors, the WOWA rank reversal phenomenon can easily be explained with an analysis of the graph of function φ. Note that in the case of equal importance weights (Fig. 1), weight v_1 is defined by an interval on a high slope segment of φ whereas both v_2 and v_3 are defined on a lower slope segment. While increasing the importance weight for η_3 one gets increased v_3 due to expanded interval. Intervals defining v_1 and v_2 are appropriately decreased. However, while v_1 is indeed decreased, v_2 is actually increased since a smaller interval is applied to a higher slope, as the expansion of $\bar{p}_3$ pushes $\bar{p}_2$ on the high slope segment of function φ.

4 Asymptotic Monotonicity

In the previous section, we have given a counterexample illustrating that rank stability may not hold for the WOWA aggregation. Now, we show that nonetheless it satisfies asymptotic monotonicity and we give required levels of importance weight change to guarantee it.

The WOWA aggregation is continuous with respect to importance weights. Therefore, it obviously fulfills the property of asymptotic monotonicity. Note that for any outcome vectors η' and η'' such that $\eta'_{i_o} < \eta''_{i_o}$ for an attribute $i_o \in I$, one gets $WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta') < WOWA_{\mathbf{w},\bar{\mathbf{p}}}(\eta'')$ with $\bar{\mathbf{p}} = \mathbf{e}_{i_o} = \lim_{\varepsilon\to\infty} \mathbf{p}(\varepsilon)$, following (10). Indeed, the following statement can be directly proven.

Proposition 3 *Let η' and η'' be outcome vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$. For any positive preferential weights $w_i \geq 0$ and any vector of importance weights $\mathbf{p}$, a sufficiently large increase of importance weight p_{i_o} with proportional decrease of other weights, following (10) with $\varepsilon > \Delta$*

$$\Delta = \max\left\{\frac{\max\limits_{i\neq i_o}(\eta'_i - \eta''_i)\max\limits_{k\in I} w_k}{(\eta''_{i_o} - \eta'_{i_o})\min\limits_{k\in I} w_k}, 0\right\} + \frac{2\max\limits_{i\in I}|\eta''_i|(\max\limits_{k\in I} w_k - \min\limits_{k\in I} w_k)}{(\eta''_{i_o} - \eta'_{i_o})\min\limits_{k\in I} w_k}$$

results in strict inequality $WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') < WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'')$.

Proof. Note that following (5), one gets

$$\begin{aligned} WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') - WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'') &= \sum_{i=1}^m \pi_i(\mathbf{p}(\varepsilon),\eta')\eta'_i - \sum_{i=1}^m \pi_i(\mathbf{p}(\varepsilon),\eta'')\eta''_i \\ &= \sum_{i=1}^m \pi_i(\mathbf{p}(\varepsilon),\eta')(\eta'_i - \eta''_i) + \sum_{i=1}^m (\pi_i(\mathbf{p}(\varepsilon),\eta') - \pi_i(\mathbf{p}(\varepsilon),\eta''))\eta''_i \end{aligned}$$

and from (6), one gets for any outcome vector η

$$mp_i(\varepsilon)\min_{k\in I} w_k \leq \pi_i(\mathbf{p}(\varepsilon),\eta) \leq mp_i(\varepsilon)\max_{k\in I} w_k,$$

$$1 - m(1-p_i(\varepsilon))\max_{k\in I} w_k \leq \pi_i(\mathbf{p}(\varepsilon),\eta) \leq 1 - m(1-p_i(\varepsilon))\min_{k\in I} w_k.$$

Hence, $|\pi_i(\mathbf{p}(\varepsilon),\eta') - \pi_i(\mathbf{p}(\varepsilon),\eta'')| \leq m(\max_{k\in I} w_k - \min_{k\in I} w_k)\min\{p_i(\varepsilon), 1-p_i(\varepsilon)\}$, $\pi_{i_o}(\mathbf{p}(\varepsilon),\eta') \geq mp_{i_o}(\varepsilon)\min_{k\in I} w_k$, $\pi_i(\mathbf{p}(\varepsilon),\eta') \leq mp_i(\varepsilon)\max_{k\in I} w_k$ for $i \neq i_o$ and

$$\begin{aligned} &WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') - WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'') \\ &\quad\leq m[\min_{k\in I} w_k p_{i_o}(\varepsilon)(\eta'_{i_o} - \eta''_{i_o}) + \max_{k\in I} w_k \sum_{i\neq i_o} p_i(\varepsilon)\max\{\max_{i\neq i_o}\eta'_i - \eta''_i, 0\}] \\ &\quad+ m(\max_{k\in I} w_k - \min_{k\in I} w_k)[(1-p_{i_o}(\varepsilon))|\eta''_{i_o}| + \sum_{i\neq i_o} p_i(\varepsilon)\max_{i\neq i_o}|\eta''_i|] \\ &\quad\leq m\min_{k\in I} w_k[p_{i_o}(\varepsilon) - (1-p_{i_o}(\varepsilon))\Delta](\eta'_{i_o} - \eta''_{i_o}). \end{aligned}$$

Thus, for large enough $\varepsilon > \Delta$ one gets $p_{i_o}(\varepsilon) = (p_{i_o} + \varepsilon)/(1+\varepsilon) > \Delta/(1+\Delta)$ and thereby $WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') < WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'')$.

Proposition 3 states that when having a WOWA optimal solution with a non satisfactory achievement for criterion i_o, one may increase the importance of this criterion, e.g., by setting new importance weights $p(\varepsilon)_{i_o} = (p_{i_o} + \varepsilon)/(1+\varepsilon)$ and $p(\varepsilon)_i = p_i/(1+\varepsilon)$ for all $i \neq i_o$. For a sufficiently large increment ε, following Proposition 3 it will exclude solutions with worse achievements for criterion i_o. However, the required amount of the weight increase for a general case, following Δ in Proposition 3 is impracticably large. It can be reduced for special types of the WOWA operators like for the case of monotonic preferential weights which is well suited for decisions under risk [10] or fair optimization [11].

Note that, following (8), we have

$$WOWA_{\mathbf{w},\mathbf{p}}(\eta') - WOWA_{\mathbf{w},\mathbf{p}}(\eta'') = \sum_{i=1}^{m} \overline{w}_k m[L(\eta', \mathbf{p}, \frac{k}{m}) - L(\eta'', \mathbf{p}, \frac{k}{m})]$$

where $\overline{w}_k$ are positive differential OWA weights defined as (9) and

$$L(\eta', \mathbf{p}, \xi) - L(\eta'', \mathbf{p}, \xi) = \max_{\mathbf{u} \in U(\mathbf{p},\xi)} \sum_{i=1}^{m} \eta'_i u_i - \max_{\mathbf{u} \in U(\mathbf{p},\xi)} \sum_{i=1}^{m} \eta''_i u_i$$

with $U(\mathbf{p}, \xi) = \{\mathbf{u} = (u_1, \ldots, u_m) : \sum_{i=1}^{m} u_i = \xi, \quad 0 \leq u_i \leq p_i \quad i \in I\}$. Hence,

$$L(\eta', \mathbf{p}, \xi) - L(\eta'', \mathbf{p}, \xi) \leq \sum_{i=1}^{m} \eta'_i \bar{u}_i(\xi) - \sum_{i=1}^{m} \eta''_i \bar{u}_i(\xi) = \sum_{i=1}^{m} (\eta'_i - \eta''_i)\bar{u}_i(\xi) \quad (13)$$

where $\bar{u}(\xi)$ is an optimal solution to the problem $\max_{\mathbf{u} \in U(\mathbf{p},\xi)} \sum_{i=1}^{m} \eta'_i u_i$.

Proposition 4 *Let η' and η'' be outcome vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$ and $\eta'_{i_o} \geq \eta'_i$ for all $i \in I$. For any positive and decreasing preferential weights $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ and any vector of importance weights $\mathbf{p}$, a sufficiently large increase of importance weight p_{i_o} with proportional decrease of other weights, following (10) with $\varepsilon > \Delta$*

$$\Delta = \max\left\{\max_{i \neq i_o} \frac{\eta'_i - \eta''_i}{\eta''_{i_o} - \eta'_{i_o}}, 0\right\}$$

results in strict inequality $WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') < WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'')$.

Proof. Applying inequality (13) to importance weights $\mathbf{p}(\varepsilon)$ one gets

$$L(\eta', \mathbf{p}(\varepsilon), \xi) - L(\eta'', \mathbf{p}(\varepsilon), \xi) \leq [\bar{u}_{i_o}(\xi) - \Delta \sum_{i \neq i_o} \bar{u}_i(\xi)](\eta'_{i_o} - \eta''_{i_o})$$

where, due to $\eta'_{i_o} \geq \eta'_i$ for all i, $\bar{u}_{i_o}(\xi) = \min\{\xi, p_{i_o}(\varepsilon)\}$ and $\bar{u}_i(\xi) \leq \min\{\xi - \bar{u}_{i_o}(\xi), p_i(\varepsilon)\}$ for all $i \neq i_o$. Hence,

$$L(\eta', \mathbf{p}(\varepsilon), \xi) - L(\eta'', \mathbf{p}(\varepsilon), \xi) \leq \begin{cases} \xi(\eta'_{i_o} - \eta''_{i_o}) & \xi \leq p_{i_o}(\varepsilon) \\ (p_{i_o}(\varepsilon) - \Delta \sum_{i \neq i_o} p_i(\varepsilon))(\eta'_{i_o} - \eta''_{i_o}) & \xi > p_{i_o}(\varepsilon) \end{cases}$$

Therefore, for a large enough $\varepsilon > \Delta$ one gets $p_{i_o}(\varepsilon) > \Delta/(1+\Delta)$ and $p_{i_o}(\varepsilon) - \Delta(1-p_{i_o}(\varepsilon)) > 0$. Thus $L(\eta', \mathbf{p}(\varepsilon), \xi) < L(\eta'', \mathbf{p}(\varepsilon), \xi)$ for any $0 < \xi \leq 1$ and, due to nonnegative differential weights $\bar{w}_k$, inequality $WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') < WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'')$ is valid.

Proposition 5 *Let η' and η'' be outcome vectors such that $\eta'_{i_o} < \eta''_{i_o}$ for a criterion $i_o \in I$. For any positive and decreasing preferential weights $w_1 \geq w_2 \geq \ldots \geq w_m > 0$ and any vector of importance weights $\mathbf{p}$, a sufficiently large increase of importance weight p_{i_o} with proportional decrease of other weights, following (10) with $\varepsilon > \Delta$*

$$\Delta = \max\left\{\max_{i \neq i_o} \frac{(\eta'_i - \eta''_i)w_1}{(\eta''_{i_o} - \eta'_{i_o})w_m}, 0\right\}$$

results in strict inequality $WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') < WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'')$.

Proof. Let $\delta = \max\{\max_{i \neq i_o}(\eta'_i - \eta''_i), 0\}/(\eta''_{i_o} - \eta'_{i_o})$. Applying inequality (13) to importance weights $\mathbf{p}(\varepsilon)$ one gets

$$\begin{aligned} WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') - & WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'') \\ & \leq \sum_{k=1}^{m} \overline{w}_k m[\bar{u}_{i_o}(\frac{k}{m}) - \delta \sum_{i \neq i_o} \bar{u}_i(\frac{k}{m})](\eta'_{i_o} - \eta''_{i_o}) \\ & \leq m[w_m p_{i_o}(\varepsilon) - \delta w_1 \sum_{i \neq i_o} p_i(\varepsilon)](\eta'_{i_o} - \eta''_{i_o}) \end{aligned}$$

since $\bar{u}_{i_o}(\frac{k}{m}) \geq 0$ for all k, $\bar{u}_{i_o}(\frac{m}{m}) = p_{i_o}(\varepsilon)$, and $\bar{u}_i(\frac{k}{m}) \leq p_i(\varepsilon)$ for all i. Thus, for a large enough $\varepsilon > \Delta$ one gets $p_{i_o}(\varepsilon) > \Delta/(1+\Delta) = \delta w_1/(\delta w_1 + w_m)$ and thereby $WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta') < WOWA_{\mathbf{w},\mathbf{p}(\varepsilon)}(\eta'')$.

5 Concluding remarks

In this paper, we have investigated the monotonicity of WOWA with respect to weight perturbations in favor of a single criterion. Contrary to intuition, there exist configurations where such an improvement in favor of a criterion i_o impact negatively the performance of the optimal solution on that criterion. This may reduce the controllability of WOWA when used as a scalarizing function in interactive exploration of feasible solutions. Hopefully, we also have established positive results showing that some controllability can be ensured for sufficiently large weight improvements. Our results show that the WOWA importance weighting mechanism alone is insufficient for effective multiple criteria preference modeling. For this purpose the WOWA aggregation should be supported by additional control parameters like aspiration levels in the reference point methods [7]. We think similar studies are worth investigating for a more general class of aggregation operators, such as Choquet integrals.

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References

1. Damiani, E., De Capitani di Vimercati, S., Samarati, P., Viviani, M.: A WOWA-based aggregation technique on trust values connected to metadata. Electronic Notes in Theoretical Computer Science. **157**(3), 131–142, (2006).
2. Larsen, H.L.: Importance weighted OWA aggregation of multicriteria queries. North American Fuzzy Information Processing Society (NAFIPS). 740–744, (1999).
3. Liu, X.: Some properties of the weighted OWA operator. IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics. **368**(1), 118–127, (2006)
4. Llamazares, B.: (2004) Simple and absolute special majorities generated by OWA operators. European Journal of Operational Research. **158**(3), 707–720, (2004)
5. Nettleton, D.F., Muñiz, J.: Processing and representation of meta-data for sleep apnea diagnosis with an artificial intelligence approach. International Journal of Medical Informatics. **63**(1-2), 77–89, (2001)
6. Ogryczak, W.: Multiple criteria optimization and decisions under risk. Control & Cybernetics. **31**, 975–1003, (2002)
7. Ogryczak, W.: Ordered weighted enhancement of preference modeling in the reference point method for multicriteria optimization. Soft Comp. **14**, 345–360, (2010)
8. Ogryczak, W., Perny, P., Weng, P.: On Minimizing Ordered Weighted Regrets in Multiobjective Markov Decision Processes. ADT. LNAI. **6992**, 190–204, (2011)
9. Ogryczak, W., Śliwiński, T.: On Optimization of the importance weighted OWA aggregation of multiple criteria. ICCSA. LNCS. **4705**, 804-817, (2007)
10. Ogryczak, W., Śliwiński, T.: On efficient WOWA optimization for decision support under risk. Int. Journal of Approximate Reasoning. **50**(6), 915–928, (2009)
11. Ogryczak, W., Wierzbicki, A., Milewski, A.: A multi-criteria approach to fair and efficient bandwidth allocation. Omega. **36**(3), 451–463, (2008)
12. Quiggin, J.: Generalized Expected Utility Theory. The Rank-Dependent Model. Kluwer Academic, Dordrecht, (1993)
13. Torra, V.: The weighted OWA operator. Int. J. Intell. Syst. **12**(2), 153–166, (1997)
14. Torra, V., Narukawa, Y.: Modeling Decisions Information Fusion and Aggregation Operators. Springer-Verlag, Berlin (2007).
15. Valls, A., Torra, V.: Using classification as an aggregation tool for MCDM. Fuzzy Sets Systems. **115**(1), 159-168, (2000)
16. Yager, R.R.: On ordered weighted averaging aggregation operators in multicriteria decision making. IEEE Trans. on Syst., Man and Cyber. **18**(1), 183–190, (1988)
17. Yager, R.R.: On the analytic representation of the Leximin ordering and its application to flexible constraint propagation. European Journal of Operational Research. **102**(1), 176–192, (1997)
18. Yager, R.R.: Including Importances in OWA Aggegations Using Fuzzy Systems Modeling. IEEE Transactions on Fuzzy Systems. **6**(2), 286–294, (1998)
19. Yager, R.R., Filev, D.P.: Essentials of Fuzzy Modeling and Control, Wiley, (1994)
20. Yager, R.R., Kacprzyk, J., Beliakov, G. (Eds.): Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice. Springer, (2011)