An Axiomatic Approach to Qualitative Decision Theory with Binary Possibilistic Utility

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Abstract. Binary possibilistic utility unifies two previously proposed qualitative decision models: optimistic and pessimistic utilities. All these decision models have been axiomatized in a von Neumann-Morgenstern setting. These axiomatizations have shown the formal similarity of these qualitative utilities and expected utility. Unfortunately in this framework, the representation of uncertainty has to be clearly assumed to be given. In a more general setting, without this restriction, optimistic and pessimistic utilities have been axiomatized à la Savage. This paper proposes a study of the axiomatics of binary possibilistic utility in a Savagean framework.

1 INTRODUCTION

Decision making under uncertainty is an important topic in AI research, whether in the process of building autonomous agents or in modeling the behavior of other agents (human or not) in the course of a multi-agent interaction, since any artificial agent can be viewed as an automated decision maker [21]. Decision theory has been first studied in economics, which explains why quantitative decision models are the most common ones and have been the most investigated. Among those is the classic model of expected utility (EU). The axiomatic works of von Neumann and Morgenstern (vNM) [26] and Savage [22] highlighted the properties of this criterion and described the situations where it should be employed. This model is known to make some strong hypotheses: uncertainty has to be represented by probability and a numeric value (utility) is assigned to each consequence of an action. These assumptions are not suitable to all kinds of problems. Indeed, in the situations where data are imprecise and/or scarce, the use of probability is sometimes debatable. Even when probability is appropriate, the assessment of the parameters of the model (probabilities and utilities) is a difficult and information demanding task.

For these reasons, it is of interest to study other decision models. In the situations where probability is not suitable, many alternatives have been proposed: theory of evidence [24], possibility theory [8], kappa rankings [25] or plausibility measures [13]. In each of these frameworks, decision models have been proposed [18, 9, 14, 4].

In AI, with its long standing tradition of symbolic problem modeling and solving, qualitative decision models ([2], [3], [19],...) have received much attention. They are very appealing as they are not information demanding (only qualitative information is required) and lead to faster computations. In this paper, we restrict ourself to qualitative decision models based on possibility theory and more specifically to binary possibilistic utility (PU) that was introduced by Giang and Shenoy [15]. This qualitative model has many interesting properties. It unifies two previously proposed qualitative utilities: optimistic and pessimistic utilities [9, 6] and can model decision behavior that neither optimistic nor pessimistic utilities can model. Moreover it is dynamically consistent and therefore can be applied in sequential decision making, especially in Markov Decision Processes [20]. Finally it can be considered the qualitative counterpart of EU. Indeed, Weng [27] has shown that a same set of axioms (complete preorder, independence, continuity and non triviality) leads to either EU or PU depending on whether uncertainty is represented by respectively a probability or a possibility distribution.

Axiomatizing a decision criterion is essential as it reveals its properties and its limits. The axiomatization of EU highlighted its restrictions [1, 12]. Such work has to be done in the qualitative setting, which encompasses logic-formalized decision problems [9]. Axiomatization can be lead in two different frameworks: in a vNM setting where the nature of uncertainty is assumed to be given here of possibilistic nature or in a Savagean setting where no such prior assumption is made. The axiomatic studies of this criterion have all been lead in a vNM framework [15, 16]. Dubois et al. [10] axiomatized optimistic and pessimistic utilities in a Savagean context. These works give new insights about the two qualitative utilities in a very general framework. Following their works, we propose here to give a Savagean axiomatization of PU. Interestingly, such works allow the nature of uncertainty as perceived by the decision maker to be revealed: possibility for optimistic utility, necessity for pessimistic utility and as shown in this paper a pair consisting of possibility and necessity for PU. Consequently, this gives some justifications for the use of non probabilistic uncertainty representation.

This paper is organized as follows. In Section 2, we recall all the necessary notions for the presentation of our results. Due to lack of space, we do not present the Savagean axiomatics for EU and its comparison with that of qualitative utilities. We refer the interested reader to [10]. Then, in Section 3, we present some properties of PU for a better understanding of this decision model. In Section 4, we present our axiomatization. Finally we conclude in Section 5.

2 POSSIBILISTIC UTILITIES

2.1 Some Definitions and Notations

In decision under uncertainty, a decision maker has to choose an action, named act, whose consequence is not known precisely. The decision problem can be described by the following elements:

- S a finite set of states.
- $X = \{x_1, \dots, x_n\}$ a finite set of consequences. Elements 0_X and 1_X denote respectively the worst and the best consequences.
- $\mathcal{F}$ a finite set of acts, i.e. functions from the set of states to the set of consequences ($\mathcal{F} = X^S$).

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A state (or state of nature) describes a possible configuration of the environment. The decision maker has to choose an act, which is a function from the set of states to the set of consequences. The decision maker's preference relation (which is simply a binary relation) over acts is denoted by $\succsim_{\mathcal{F}}$, which reads "at least as good as".

Uncertainty follows from the fact that the decision maker does not know precisely in which state he is. Uncertainty is then defined as uncertainty about the actual state of nature. Events are simply sets of states. The set of all events is denoted 2^S. For two events A and B, $A \succsim_S B$ means that A is judged at least as plausible than B by the decision maker. The complement of an event $A \in 2^S$ is denoted $\overline{A}$.

For two acts f and g, an event A, fAg denotes the act consisting in doing f if event A and doing g otherwise, i.e. $\forall s \in A, fAg(s) = f(s)$ and $\forall s \in \overline{A}, fAg(s) = g(s)$. The constant act equal to $x \in X$ everywhere is denoted $\mathbf{x}$. Relation $\succsim_{\mathcal{F}}$ restricted to constant acts induces a preference relation over consequences, denoted $\succsim_X$.

For any relation $\succsim$, $\succ$ and $\sim$ denote respectively the asymmetric and the symmetric part of $\succsim$.

A finite qualitative scale L on which consequences and uncertainty can be measured is defined. Its lowest and its greatest elements are written respectively by 0_L and 1_L. The order on L is denoted by $\geq_L$. Operators $\max$ and $\min$ on scale L are respectively written $\vee$ and $\wedge$. The order reversing involution in L is denoted by n, i.e. $n(0_L) = 1_L, n(1_L) = 0_L$ and $\forall \lambda, \lambda' \in L, \lambda \geq_L \lambda' \Rightarrow n(\lambda) \leq_L n(\lambda')$. By abuse of notation, in the paper, n will denote any such one-to-one mapping that reverses a preordered scale.

A *basic utility assignment* is a mapping that associates a value in a scale to each consequence in X. A *capacity* $\sigma : 2^S \to L$ is a very general measure of uncertainty that satisfies:

- $\sigma(\emptyset) = 0_L$
- $\sigma(S) = 1_L$
- $\forall A, B \in 2^S, A \subseteq B \Rightarrow \sigma(A) \leq_L \sigma(B)$

As examples, probability, possibility, belief functions, kappa rankings and plausibility measures are all capacities. We recall the definitions of a possibility distribution. A capacity is a possibility distribution (denoted π) iff it additionally satisfies $\forall A \in 2^S, \pi(A) = \bigvee_{s \in A} \pi(s)$. Distribution π can be viewed as an encoding of the ranking of the states in terms of plausibility. It can be interpreted as follows: $\pi(s) = 0_L$ means that state s is impossible, $\pi(s) = 1_L$ means that s is totally possible, when $0_L <_L \pi(s) <_L 1_L$, s is only possible to some extent, i.e. there are states more possible than s. We assume that there exists at least one state of possibility 1_L and of course, there could be several ones.

We introduce a few axioms on the preorder over events.

A1 Relation $\succsim_S$ over 2^S is a total preorder
A2 $S \succ_S \emptyset$
A3 $\forall A \in 2^S, A \succsim_S \emptyset$

These axioms corresponds to the properties of a capacity. The following theorem can then be stated [11]:

Theorem 1. *If $\succsim_S$, a preorder over states, satisfies axioms* **A1**, **A2** *and* **A3** *then it exists a finite qualitative scale $(L, \geq_L)$ and a capacity $\sigma : 2^S \to L$ such that $A \succsim_S B \Leftrightarrow \sigma(A) \geq_L \sigma(B)$.*

Adding the following axiom, the capacity becomes a possibility distribution [5].

POS $\forall A, B, C \in 2^S, A \succsim_S B \Rightarrow A \cup C \succsim_S B \cup C$.

Dually the following axiom characterizes a necessity distribution.

NES $\forall A, B, C \in 2^S, A \succsim_S B \Rightarrow A \cap C \succsim_S B \cap C$.

2.2 Sugeno Integral

Before introducing the qualitative utilities, we present the Sugeno integral, which is a very general decision criterion. It can be seen as the qualitative counterpart of Choquet expected utility (CEU) [17, 23]. Here, the basic utility assignment is written $\mu : X \to L$.

The utility of an act defined as a Sugeno integral is defined:

$$U_S(f) = \bigvee_{x \in X} \big(\mu(x) \wedge \sigma(F_x)\big) \qquad (1)$$

where $F_x = \{s \in S : f(s) \succsim_X x\}$.

To understand this formula, Dubois and Prade [7] showed that $U_S(f)$ is the median of the utilities and uncertainty values in the following set: $\{\mu(x) : x \in X\} \cup \{\sigma(F_x) : x \in X, x \succ_X 0_X\}$. This highlights the similarity with the quantitative models (CEU or EU), which computes a weighted average.

The axioms used for the axiomatization of Sugeno integrals are:

Sav1 Preference relation $\succsim_{\mathcal{F}}$ over $\mathcal{F}$ is a total preorder.
WS3 $\forall x, y \in X, x \succsim_X y \Rightarrow \forall A \in 2^S, \forall h \in \mathcal{F}, xAh \succsim_{\mathcal{F}} yAh$
Sav5 $\exists x, x' \in X, x \succ_X x'$
RCD $\forall f, g \in \mathcal{F}, \forall x \in X, g \succ_{\mathcal{F}} f$ and $\mathbf{x} \succ_{\mathcal{F}} f \Rightarrow g \wedge \mathbf{x} \succ_{\mathcal{F}} f$
RDD $\forall f, g \in \mathcal{F}, \forall x \in X, f \succ_{\mathcal{F}} g$ and $f \succ_{\mathcal{F}} \mathbf{x} \Rightarrow f \succ_{\mathcal{F}} g \vee \mathbf{x}$

Axiom **Sav1** states that any two acts can be compared and the preference relation is transitive. Axiom **WS3** declares that when there is a preference between two consequences, this preference can not reverse if considering two acts giving these consequences on a certain event and the same consequences on the complementary event. Axiom **Sav5** is enforced in order to avoid triviality. Axiom **RCD** states that lowering the consequences of an act f, which is preferred to an act g, to a constant consequence that is also preferred to g still yields a better act than g. Finally axiom **RDD** is the dual of **RCD**. It says that an act f is preferred to an act g even if the worst consequences of g are improved to a constant consequence that is less preferred than f. By way of information, EU verifies neither **RCD** nor **RDD** [10].

Sugeno integrals [10] are characterized by:

Theorem 2. *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies* **Sav1**, **WS3**, **Sav5**, **RCD** *and* **RDD** *then there is a finite qualitative scale $(L, \geq_L)$, a basic utility assignment $\mu : X \to L$ and a capacity $\sigma : 2^S \to L$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow U_S(f) \geq_L U_S(f')$.*

2.3 Optimistic and Pessimistic Utilities

We now present optimistic and pessimistic utilities, axiomatized in a von Neumann-Morgenstern [6] and a Savagean [10] settings. In this framework, the basic utility assignment is denoted $v : X \to L$.

Then optimistic and pessimistic utilities are functions from $\mathcal{F}$ to L. Optimistic utility writes:

$$U^+(f) = \bigvee_{s \in S} \big(\pi(s) \wedge v(f(s))\big) \qquad (2)$$

and pessimistic utility writes:

$$U^-(f) = \bigwedge_{s \in S} \big(n(\pi(s)) \vee v(f(s))\big) \qquad (3)$$

Optimistic utility is a Sugeno integral with the capacity taken as a possibility distribution. It is a generalization of the maximax criterion axiomatized by Brafman and Tennenholtz [3]. And pessimistic utility is a Sugeno integral when the capacity is a necessity distribution. It is a generalization of the maximin criterion [3].

To enforce this utility, axiom **RDD** needs to be replaced by a stronger axiom:

OPT $\forall f, g \in \mathcal{F}, \forall A \in 2^S, f \succ_{\mathcal{F}} fAg \Rightarrow gAf \succsim_{\mathcal{F}} f$

This axiom says that if in a certain event, changing the consequences of an act yields a strictly less preferred one then in the complementary event, changing its consequences cannot yield a strictly less preferred act. It reveals a particular property of optimistic utility. If an act can be strictly downgraded in a certain event, this event is judged plausible enough to attract all the attention of the decision maker and what happens on the complementary event is not relevant.

The representation theorem for the optimistic utility [10] states:

Theorem 3. *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies* **Sav1**, **WS3**, **Sav5**, **RCD** *and* **OPT** *then there is a finite qualitative scale $(L, \geq_L)$, a basic utility assignment $v : X \to L$ and a possibility distribution $\pi : 2^S \to L$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow U^+(f) \geq_L U^+(f')$.*

For the pessimistic utility, axiom **RCD** is to be replaced by:

PES $\forall f, g \in \mathcal{F}, \forall A \in 2^S, fAg \succ_{\mathcal{F}} f \Rightarrow f \succsim_{\mathcal{F}} gAf$

This axiom, which is the dual of **OPT** states that if an act is stricly improved in a certain event then it can not be strictly improved in the complementary event.

And the pessimistic utility [10] is characterized by:

Theorem 4. *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies* **Sav1**, **WS3**, **Sav5**, **RDD** *and* **PES** *then there is a finite qualitative scale $(L, \geq_L)$, a basic utility assignment $v : X \to L$ and a possibility distribution $\pi : 2^S \to L$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow U^-(f) \geq_L U^-(f')$.*

2.4 Binary Possibilistic Utility

Binary possibilistic utility (PU) introduced by Giang and Shenoy [15] unifies the two previous qualitative decision models. It takes values in a particular scale L_2, which is a binary scale defined by the following set of pairs:

$$L_2 = \{\langle \lambda, \mu \rangle : \lambda, \mu \in L, \lambda \vee \mu = 1_L\}.$$

Due to constraint $\lambda \vee \mu = 1_L$, scale L_2 is in fact monodimensional (see fig. 1). A pair in L_2 can be interpreted as a binary possibility

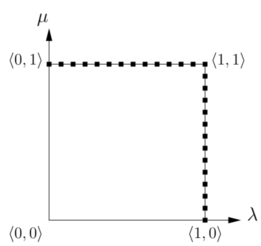

Figure 1. Binary utility scale L_2.

distribution over two consequences : the best and the worst ones. The first element of the pair would be the possibility to get the best consequence and the second would be the possibility to get the worst consequence. Then a natural order relation[2] can be defined on L_2:

$$\langle \lambda, \mu \rangle \geq_{L_2} \langle \lambda', \mu' \rangle \iff (\lambda \geq_L \lambda' \text{ and } \mu \leq_L \mu').$$

[2] This order is expressed in a simpler form than in [15, 16]. It can be easily checked that the two forms are equivalent.

Indeed, the decision maker naturally prefers the highest possibility of getting the best outcome and the lowest possibility of getting the worst one.

Here, the basic utility assignment is denoted $u : X \to L_2$. Remark that it takes values in the binary scale. Uncertainty represented by a possibility distribution is still measured on L.

Operator $\vee$ is extended as an operator on $L_2 \times L_2$ as follows:

$$\langle \lambda, \mu \rangle \vee \langle \lambda', \mu' \rangle = \langle \lambda \vee \lambda', \mu' \vee \mu' \rangle.$$

Remark that this operator is not operator $\max$ on L_2, which would be computed as $\max(\langle \lambda, \mu \rangle, \langle \lambda', \mu' \rangle) = \langle \lambda \vee \lambda', \mu \wedge \mu' \rangle$.

Operator $\wedge$ is extended as an operator on $L \times L_2$:

$$\lambda' \wedge \langle \lambda, \mu \rangle = \langle \lambda' \wedge \lambda, \lambda' \wedge \mu \rangle.$$

In the same way as operator $\max$, operator $\min$ on L_2 is given by $\min(\langle \lambda, \mu \rangle, \langle \lambda', \mu' \rangle) = \langle \lambda \wedge \lambda', \mu \vee \mu' \rangle$.

PU is then defined as a function from $\mathcal{F}$ to L_2:

$$PU(f) = \bigvee_{s \in S} \big(\pi(s) \wedge u(f(s))\big). \tag{4}$$

As noted in [15], when the basic utility has the following restricted form: $\forall x \in X, \exists \lambda \in L, u(x) = \langle \lambda, 1_X \rangle$ then PU is an optimistic utility. Dually when we have $\forall x \in X, \exists \mu \in L, u(x) = \langle 1_X, \mu \rangle$, PU is a pessimistic utility. Finally we remark that compared to PU, optimistic and pessimistic utilities exploit only half of scale L_2.

3 STUDY OF THE PU MODEL

3.1 Properties of PU

We assume here that no two consequences in X are equivalent. This is not a restrictive condition as if it were not the case, one could work on the equivalence classes of X instead of X. This condition entails that X is a preordered scale. Let $n \circ f$ be the act that is defined by $\forall s \in S, (n \circ f)(s) = n(f(s))$. Then the *dual* of a preference relation $\succsim_{\mathcal{F}}$ over $\mathcal{F}$, denoted $\succsim_{\mathcal{F}}{}^T$, is defined by $f \succsim_{\mathcal{F}}{}^T g \Leftrightarrow n \circ g \succsim_{\mathcal{F}} n \circ f$. Obviously the dual of the dual of a preference relation is the relation itself. By extension, we say that a decision model is the dual of another one if the preference relation defined by the former decision model is the dual of the relation defined by the latter. Pessimistic utility is the dual of optimistic utility.

A preference relation is said to be *autodual* if and only if $\succsim_{\mathcal{F}}{}^T = \succsim_{\mathcal{F}}$. This means that for two acts f and g, $f \succsim_{\mathcal{F}} g \Leftrightarrow n \circ g \succsim_{\mathcal{F}} n \circ f$. Again, by extension, a decision model is said to be autodual if the induced preference relation is autodual. Intuitively, it means that the decision model when stating a preference between two acts looks at how well and how bad these two acts do. Any decision model that is not autodual does not use all the information at its disposal since it could be possible to exploit the dual of the decision model to yield a refined criterion. In this sense a rational decision maker would rather use an autodual decision model.

As the first and second part of PU are respectively of the form of an optimistic utility and a pessimistic one, PU induces an autodual preference relation over acts. EU is another example of such an autodual decision model. Once again, similarities between the two criteria can be emphasized.

When the basic utility assignment is valued in L_2 and uncertainty is measured by a possibility distribution, Eqs. 1 and 4 are equivalent.

Proposition 1. $PU(f) = U_S(f)$

Proof. We prove the result recursively on n the number of different consequences of f. Let $f = x_1 A_1 x_2 A_2 \dots x_n A_n$ where $x_1 \succ_X x_2 \succ_X \dots \succ_X x_n$. We want to prove that $PU(f) = U_S(f)$. Assume for $n = 2$ that $f = xAy$ with $x \succ_X y$. Then $PU(f) = \langle (u_1(x) \wedge \pi(A)) \vee (u_1(y) \wedge \pi(\overline{A})), (u_2(x) \wedge \pi(A)) \vee (u_2(y) \wedge \pi(\overline{A})) \rangle$.

$$\begin{aligned} U_S(f) &= \max(\langle u_1(y), u_2(y) \rangle, \\ &\quad \min(\langle u_1(x), u_2(x) \rangle, \langle \pi(A), \pi(\overline{A}) \rangle)) \\ &= \max(\langle u_1(y), u_2(y) \rangle, \langle u_1(x) \wedge \pi(A), u_2(x) \vee \pi(\overline{A}) \rangle) \\ &= \langle u_1(y) \vee (u_1(x) \wedge \pi(A)), u_2(y) \wedge (u_2(x) \vee \pi(\overline{A})) \rangle \\ &= \langle u_1(y) \vee (u_1(x) \wedge \pi(A)), u_2(x) \vee (u_2(y) \wedge \pi(\overline{A})) \rangle \end{aligned}$$

Now if $\pi(A) = 1_L$ then $PU(f) = \langle u_1(x) \vee (u_1(y) \wedge \pi(\overline{A})), u_2(x) \vee (u_2(y) \wedge \pi(\overline{A})) \rangle = \langle u_1(x), u_2(x) \vee (u_2(y) \wedge \pi(\overline{A})) \rangle$ and $U_S(f) = \langle u_1(y) \vee u_1(x), u_2(x) \vee (u_2(y) \wedge \pi(\overline{A})) \rangle = PU(f)$. In the same way, we can check that if $\pi(\overline{A}) = 1_L$ then $PU(f) = U_S(f)$. Now assume the property is true for n and prove that it is true for $n+1$. We have $f = x_1 A_1 x_2 A_2 \dots x_{n+1} A_{n+1}$ where $x_1 \succ_X x_2 \succ_X \dots \succ_X x_{n+1}$. If $\pi(A_1) < 1_L$ apply the recursion assumption on $g = x_2 A_2 \dots x_{n+1} A_{n+1}$. Otherwise, apply the recursion assumption on $g = x_1 A_1 \dots x_n A_n \cup A_{n+1}$. We get $PU(g) = U_S(g)$. Then apply the case $n = 2$ to get the result. □

The previous result is not surprising. Being the unification of optimistic and pessimistic utilities, which are two Sugeno integrals, PU is naturally also a Sugeno integral. Indeed PU satisfies all the required conditions of Theorem 2.

Proposition 2. *PU satisfies* **Sav1**, **WS3**, **Sav5**, **RCD** *and* **RDD**.

Consequently, in the PU model, uncertainty is not measured by a possibility distribution, as it could be at first thought, but by the pair of a possibility distribution and its associated necessity distribution. Indeed as noted in Proposition 1, uncertainty is measured by the pair $\langle \pi(A), \pi(\overline{A}) \rangle$. We recall that the necessity distribution that is associated to a possibility distribution is defined by $N(A) = n(\pi(\overline{A}))$.

Before going on with the properties of the underlying uncertainty measure, we introduce the axiom of weak independence. This is a weakening of the axiom of the sure thing principle used by Savage for EU.

$$\textbf{WI} \quad \left. \begin{array}{l} \forall f, g, h, h' \in \mathcal{F} \\ \forall A \in 2^S \end{array} \right\} fAh \succ_{\mathcal{F}} gAh \Rightarrow fAh' \succsim_{\mathcal{F}} gAh'$$

This axiom is crucial in dynamic decision making and ensures that dynamic consistency holds. EU of course fulfills the sure thing principle and **WI**. Axiom **WI** is satisfied by PU guaranteeing that this criterion is dynamically consistent.

3.2 Properties of the Underlying Uncertainty Measure in PU

The properties of duality and autoduality can be also stated on the preorder over events. The *dual* of $\succsim_S$ is the relation ${\succsim_S}^T$ defined by $\forall A, B \in 2^S, A {\succsim_S}^T B$ iff $\overline{B} \succsim_S \overline{A}$. Preorder $\succsim_S$ is said to be *autodual* iff for two events A and B, $A \succsim_S B \Leftrightarrow \overline{B} \succsim_S \overline{A}$. This means that when A is more plausible than B, the complement of B is more plausible than that of A. Duality and autoduality can be naturally extended to uncertainty measures. As previously an uncertainty measure that is not autodual does not use all the information since it could be refined with its dual uncertainty measure. Therefore autodual uncertainty measure should be exploited when possible. A probability distribution is an example of such a measure.

The underlying uncertainty measure σ used in PU is said to be *autodual possibilistic*. It is defined by $\sigma : 2^S \to L_2, \forall A \in 2^S, \sigma(A) = \langle \pi(A), \pi(\overline{A}) \rangle$, which is obviously an autodual measure.

Some definitions are needed in order to characterize such capacities. For any event D, we write $D^{\prec_S}$ for the set of events: $\{A \in 2^S : A \prec_S D\}$ and $D^{\precsim_S}$ for the set of events: $\{A \in 2^S : A \precsim_S D\}$. They are respectively the set of events that are less plausible than D and its complement.

Now we introduce two axioms entailing that the capacity is an autodual possibilistic capacity.

AD^S Relation $\succsim_S$ over 2^S is autodual.
POS' $\exists D, S \succ_S D \succ_S \emptyset$ such that
$\forall A, B, C \in D^{\prec_S}, A \succsim_S B \Rightarrow A \cup C \succsim_S B \cup C$

Obviously axiom **AD**^{S} is required if we want an autodual capacity. Axiom **POS'** roughly states that the set of events can be divided into two parts such that on the less plausible side, $\succsim_S$ satisfies axiom **POS**. Together axioms **POS'** and **AD**^{S} give the dual property:

NES' $\exists D, S \succ_S D \succ_S \emptyset$ such that
$\forall A, B, C \in D^{\succsim_S}, A \succsim_S B \Rightarrow A \cup C \succsim_S B \cup C$

Axioms **POS'** and **NES'** are of course fulfilled by autodual possibilistic capacities. If there are two states s, s' that have possibility 1_L then event D is such that its uncertainty value equals $\langle 1_L, 1_L \rangle$ (with $s \in D$ and $s' \in \overline{D}$). Otherwise, let s be the state of possibility 1_L and s' the most plausible state after s. Then D is such that $\sigma(D)$ equals $\langle 1_L, \pi(s') \rangle$.

Now autodual possibilistic capacities can be characterized:

Theorem 5. *A capacity σ satisfies axioms* **POS'** *and* **AD**^{S} *if and only if it is an autodual possibilistic capacity.*

Proof. First take D from axiom **POS'**. In the following, we write $\mathbf{s}$ for the set consisting only of state s. Remark that at least one event $\mathbf{s}$ is more plausible than $\emptyset$. Otherwise, if for all state s, $\mathbf{s} \sim_S \emptyset$, by **POS'**, set S would be equivalent to $\emptyset$.

Choose a member in each equivalence class of S. We have $\mathbf{s_1} \succsim_S \mathbf{s_2} \succsim_S \dots \succsim_S \mathbf{s_n}$. Now each equivalence class of $2^S \cap D^{\prec_S}$ is equivalent to one of the $\mathbf{s_i}$ by **POS'**.

We have $\mathbf{s_1} \succsim_S D$. Otherwise, by **POS'**, set S would be less plausible than D. By contradiction, assume that $\mathbf{s_1} \succ_S D$. Then $\mathbf{s_1}$ is not in D by monotony of σ. Applying **AD**^{S} with **POS'**, we get **NES'**. Then with $A = \mathbf{s_1}, B = D, C = \overline{D}$, $A \succsim_S B$ implies $A \cap C = \emptyset \succsim_S D = B \cap C$, which is a contradiction. Then $\mathbf{s_1} \sim_S D$.

A possibility distribution $\pi : D^{\prec_S} \to L \backslash \{1_L\}$ can be constructed by **POS'**. Define $\sigma : D^{\prec_S} \to L_2$ by $\forall A \in D^{\prec_S}, \sigma(A) = \langle \pi(A), 1_L \rangle$.

Consider the case where there is another state $s \neq s_1$ such that $\mathbf{s} \sim_S \mathbf{s_1}$. Let A be an event containing s_1 but not s. By monotony of σ, $\mathbf{s_1} \precsim_S A$. By contradiction, assume $\mathbf{s} \sim_S \mathbf{s_1} \prec_S A$. Applying **NES'** with $A, B = \mathbf{s}, C = \mathbf{s}$, we get $\mathbf{s} \prec_S A \cap \mathbf{s} = \emptyset$, which is a contradiction. Then any event containing some states equivalent to $\mathbf{s_1}$ but not all those states is equivalent to $\mathbf{s_1}$. Consider the event B containing all those states. The complement of B is the event containing all the states that are less plausible than $\mathbf{s_1}$. Then by **AD**, $B \succ_S \mathbf{s_1}$. Assume there is k equivalence classes in $D^{\prec_S}$. Then 2^S has $2k+1$ equivalence classes. Let $\sigma(D) = \langle 1_L, 1_L \rangle$.

Now consider the opposite case where there is only one state s_1. The complement of $\mathbf{s_1}$ is the event containing all the states that are less plausible than $\mathbf{s_1}$. Then 2^S has $2k$ equivalence classes. In both cases, by **AD**, σ is extended on $D^{\succsim_S}$. □

Remark that the previous theorem could have been stated with axiom **NES'** in place of **POS'**.

4 Axiomatics

For a Sugeno integral to be a binary possibilistic utility, the only condition is to ensure that the capacity is an autodual possibilistic capacity.

For any act $h \in \mathcal{F}$, we write $h^{\prec_\mathcal{F}}$ for the set of acts $\{f \in \mathcal{F} : f \prec_\mathcal{F} h\}$ and $h^{\succsim_\mathcal{F}}$ for $\{f \in \mathcal{F} : f \succsim_\mathcal{F} h\}$. They are respectively the set of acts that are less preferred than h and its complement.

The following axioms are required to state our theorem:

AD $\forall A, B \in 2^S, \mathbf{1_X} A \mathbf{0_X} \succsim_\mathcal{F} \mathbf{1_X} B \mathbf{0_X} \Leftrightarrow \mathbf{1_X} \overline{B} \mathbf{0_X} \succsim_\mathcal{F} \mathbf{1_X} \overline{A} \mathbf{0_X}$

OPT' $\exists h \in \mathcal{F}$, $\mathbf{1_X} \succ_\mathcal{F} h \succ_S \mathbf{0_X}$ such that $\forall f, g \in h^{\succsim_\mathcal{F}}, \forall A \in 2^S, f \succ_\mathcal{F} fAg \Rightarrow gAf \succsim_\mathcal{F} f$

Axiom **AD** is a reformulation of axiom **AD**^{S} in the context of acts. Axiom **OPT'** states that the set of acts can be divided into two parts, where the set of less preferred acts satisfies axiom **OPT**.

The following representation theorem for PU can be stated.

Theorem 6. *If $\succsim_\mathcal{F}$, a preorder over $\mathcal{F}$, satisfies* **Sav1**, **WS3**, **Sav5**, **RCD**, **RDD**, **OPT'** *and* **AD** *then there is a finite qualitative scale $(L, \geq_L)$, a basic utility assignment $u : X \rightarrow L_2$ and a possibilistic distribution $\pi : 2^S \rightarrow L$ such that $f \succsim_\mathcal{F} f' \Leftrightarrow PU_S(f) \geq_{L_2} PU_S(f')$.*

Proof. Axioms **Sav1**, **WS3**, **Sav5** and **OPT'** implies axiom **POS'**. See Theorem 4 of [11]. Then the result follows from Proposition 1 and Theorems 2 and 5. □

What is remarkable in this result is that although PU is an autodual criterion, the initial preference relation over acts does not have to be autodual. Indeed only the autoduality for the representation of uncertainty is required and no structure is assumed over the set of consequences X. This shows that PU could be applied even when the consequences are not completely symmetrical around a middle point.

With the following axiom of autoduality of the preference relation over acts, a simplified theorem can be formulated:

AD+ Preference relation $\succsim_\mathcal{F}$ over $\mathcal{F}$ is autodual

Theorem 7. *If $\succsim_\mathcal{F}$, a preorder over $\mathcal{F}$, satisfies* **Sav1**, **WS3**, **Sav5**, **RCD**, **OPT'** *and* **AD+** *then there is a finite qualitative scale $(L, \geq_L)$, a basic utility assignment $u : X \rightarrow L_2$ and a possibilistic distribution $\pi : 2^S \rightarrow L$ such that $f \succsim_\mathcal{F} f' \Leftrightarrow PU_S(f) \geq_{L_2} PU_S(f')$.*

Proof. Axiom **RDD** follows from **RCD** and **AD+**. Axiom **AD+** implies **AD**. Then the previous theorem can be applied. □

Compared to Theorem 6, this theorem gives a better picture of the properties of PU. Thanks to the autoduality of PU, in the two previous theorem, axiom **OPT'** could be replaced by **PES'**.

5 CONCLUSION

In this paper we gave an axiomatization à la Savage for the binary possibilistic utility, which is the qualitative counterpart of expected utility. This criterion enjoys many properties similar to its quantitative counterpart: autoduality of the underlying uncertainty measure, autoduality of the decision criterion and dynamic consistency. Moreover being the unification of optimistic and pessimistic utilities, PU can model different decision attitudes (by shifting utilities of consequences) as can expected utility with convex and concave utilities.

However as a qualitative model, it sometimes lacks decisiveness. In [27], a new qualitative model, refined binary possibilistic utility, which is an improvement over PU has been proposed. As a future work, a Savagean axiomatic study could be carried on with this refined decision model in order to unveil its properties.

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