Axiomatic Foundations of Generalized Qualitative Utility

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Abstract. The aim of this paper is to provide a unifying axiomatic justification for a class of qualitative decision models comprising among others optimistic/pessimistic qualitative utilities, binary possibilistic utility, likelihood-based utility, Spohn's disbelief function-based utility. All those criteria that are instances of Algebraic Expected Utility have been shown to be counterparts of Expected Utility thanks to a unifying axiomatization in a von Neumann-Morgenstern setting when non probabilistic decomposable uncertainty measures are used. Those criteria are based on $(\oplus, \otimes)$ operators, counterpart of $(+, \times)$ used by Expected Utility, where $\oplus$ is an idempotent operator and $\otimes$ is a triangular norm. The axiomatization is lead in the Savage setting which is a more general setting than that of von Neumann-Morgenstern as here we do not assume that the uncertainty representation of the decision-maker is known.

1 Introduction

The study of decision models is an important and necessary step in the task of building autonomous agents and support systems. Indeed, in automatic decision-making, an agent has to act "rationally" while in human assistance systems, to be able to help, an agent has to model human preferences. For both cases, decision theory can answer the critical question of which decision model to implement.

Decision models are generally axiomatically studied. An axiomatization reveals the exact properties of a decision model and shows what kind of preferences it can or can not describe. For instance, the axiomatization of Expected Utility [1] allowed its limitations to be shown [2, 3]. This axiomatic work can be lead in two different settings. In a von Neumann-Morgenstern (vNM) setting, the uncertainty representation is assumed to be known while in a Savage setting, this assumption is relaxed. The second one is thus more general and interestingly, in this setting, the uncertainty representation can be revealed from the decision-maker's preferences.

Although Expected Utility (EU) is used in many applications, this quantitative model is sometimes difficult to implement as one needs to elicit precise utilities and probabilities. When this information is not available or when it is too costly to elicit, more qualitative decision models can be preferred. Besides

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as previously underlined by [4], probability is not always convenient to represent situations of partial ignorance. Possibility Theory can be considered in those situations and it is particularly justified in case of representing human beliefs as shown in psychological experiments lead by [5].

In this paper, we propose axiomatizations in a Savage setting for a class of decision models using possibilistic uncertainty representations: Generalized Qualitative Utilities, which have been proposed by [6]. Instances of this class are among others optimistic/pessimistic utilities [7], binary possibilistic utility [8], likelihood-based utility [9], Spohn disbelief function-based utility [10]. All those models have been axiomatized in a vNM setting and some of them have been axiomatized in a Savage framework [11, 12].

This work allows us to justify axiomatically previously proposed models in a very general way and to unify them under a new decision model, generalized Sugeno integral. This unifying approach shows the similarities and the differences of all models belonging to the class of generalized qualitative utilities. This paper builds on the axiomatization proposed by [11].

In the next section, we give the used notations and recall the definitions of the many utilities that we work on. In section 3, we introduce generalized Sugeno integral and give its axiomatization. Building from this, we are then able to propose representation theorems of generalized qualitative utility and optimistic/pessimistic versions of likelihood based-utility. In section 4, we extend our results to binary utilities allowing us to provide an axiomatization of (binary) likelihood based-utility. Finally we conclude in the last section and show graphically how all those models are related to each other.

2 Decision Models

2.1 Definitions and Notations

In a problem of decision under uncertainty, a decision-maker has to choose an action, named act in the Savage setting, whose consequence is not known precisely. The decision problem can be formalized by defining: S a set of states; $X = \{x_1, \ldots, x_n\}$ a finite set of decreasingly ordered consequences (Elements $0_X = x_1$ and $1_X = x_n$ denote respectively the worst and the best consequences), $\mathcal{F}$ a finite set of acts, i.e., functions from the set of states to the set of consequences ($\mathcal{F} = X^S$). A state (or state of nature) describes a possible configuration of the environment. Uncertainty is defined as uncertainty about the actual state of nature. As the decision-maker does not know exactly in which state he is, he is uncertain about the consequences of his actions.

The decision maker's preference relation (which is simply a binary relation) over acts $\mathcal{F}$ is denoted by $\succsim_{\mathcal{F}}$, which reads "at least as good as". For any relation $\succsim$, $\succ$ and $\sim$ denote respectively the asymmetric and the symmetric part of $\succsim$.

For two acts f and g, an event A, fAg denotes the conditional act consisting in doing f if event A occurs and doing g otherwise, i.e., $\forall s \in A, fAg(s) = f(s)$ and $\forall s \in \overline{A}, fAg(s) = g(s)$. The constant act equal to $x \in X$ everywhere is denoted in bold **x**. Relation $\succsim_{\mathcal{F}}$ restricted to constant acts induces a preference relation over consequences, denoted $\succsim_X$. Events, which are simply sets of states form a sigma-algebra, denoted by E. For an event A, binary act $1_X A 0_X$ is an act giving the best consequence 1_X when A occurs and the worst outcome 0_X otherwise. Preference relation $\succsim_{\mathcal{F}}$ restricted on those binary acts defines a plausibility relation over events, $\succsim_S$. Indeed for two events A and B, $1_X A 0_X \succsim_{\mathcal{F}} 1_X B 0_X$ means the decision-maker judges A at least as plausible as B. This is written $A \succsim_S B$. The complement of an event $A \in E$ is denoted $\overline{A}$.

The scale on which uncertainty (and sometimes preference) is measured is denoted by L which is totally ordered by $\geq_L$. Its lowest and greatest elements are denoted respectively by 0_L and 1_L. Operators max and min on scale L are respectively written $\vee$ and $\wedge$. The order reversing involution in L is denoted by n, i.e., $n(0_L) = 1_L, n(1_L) = 0_L$ and $\forall \lambda, \lambda' \in L, \lambda \geq_L \lambda' \Rightarrow n(\lambda) \leq_L n(\lambda')$. A *capacity* $\sigma : E \to L$ is a very general uncertainty representation. It is a measure that satisfies: $\sigma(\emptyset) = 0_L$, $\sigma(S) = 1_L$ and $\forall A, B \in E, A \subseteq B \Rightarrow \sigma(A) \leq_L \sigma(B)$. Probability, possibility, belief functions, kappa rankings and plausibility measures are all examples of capacities. We recall the definition of a possibility distribution. A capacity is a possibility distribution (denoted π) iff it additionally satisfies $\forall A \in E, \pi(A) = \bigvee_{s \in A} \pi(s)$. Distribution π can be viewed as an encoding of the ranking of the states in terms of plausibility. It can be interpreted as follows: $\pi(s) = 0_L$ means that state s is impossible, $\pi(s) = 1_L$ means that s is totally possible, when $0_L <_L \pi(s) <_L 1_L$, s is only possible to some extent, i.e., there are states more possible than s. We assume that there exists at least one state of possibility 1_L and of course, there could be several ones. A *basic utility assignment* which represents the decision-maker's preferences on consequences is a mapping that associates a value in a scale to each consequence in X. When scale L is used, a basic utility assignment is denoted $\mu : X \to L$. For clarity, we will change notations when the scale is different.

2.2 Qualitative Utilities

Optimistic/Pessimistic Utilities. We now present optimistic and pessimistic utilities, which have already been axiomatized in a vNM and a Savage settings [7, 11]. For these criteria, uncertainty is assumed to be represented by a possibility distribution. Those qualitative utilities are functions from $\mathcal{F}$ to L. Optimistic utility U^+ and pessimistic utility U^- are defined by:

$$U^+(f) = \bigvee_{s \in S} \left(\pi(s) \wedge \mu(f(s))\right) \qquad U^-(f) = \bigwedge_{s \in S} \left(n(\pi(s)) \vee \mu(f(s))\right)$$

They are generalizations of respectively the maximax and the maximin criteria, which have been axiomatized by [13]. Using order reversing function $n : X \to X$, let $n \circ f$ be the act that is defined by $\forall s \in S, (n \circ f)(s) = n(f(s))$. Then the *dual* of a preference relation $\succsim_{\mathcal{F}}$ over $\mathcal{F}$, denoted ${\succsim_{\mathcal{F}}}^T$, is defined by $f {\succsim_{\mathcal{F}}}^T g \Leftrightarrow n \circ g \succsim_{\mathcal{F}} n \circ f$. Obviously the dual of the dual of a preference relation is the relation itself. By extension, we say that a decision model is the dual of another one if the preference relation defined by the former decision model is the dual of the relation defined by the latter. Pessimistic utility is the dual of optimistic utility.

Binary Possibilistic Utility. Binary[1] possibilistic utility (PU) introduced by [8] unifies the two previous qualitative decision models. It takes values in a particular scale L_2, which is a binary scale defined by the following set:

$$L_2 = \{\langle \lambda, \mu \rangle : \lambda, \mu \in L, \lambda \vee \mu = 1_L\}.$$

The scale is binary in the sense that a value of this scale is either in interval $[\langle 1_L, 0_L \rangle, \langle 1_L, 1_L \rangle]$ or in interval $[\langle 0_L, 1_L \rangle, \langle 1_L, 1_L \rangle]$. A pair in L_2 can be interpreted as a binary possibility distribution over two consequences : the best and the worst ones. The first element of the pair would be the possibility to get the best consequence and the second would be the possibility to get the worst consequence. Then a natural order relation can be defined on L_2:

$$\langle \lambda, \mu \rangle \geq_{L_2} \langle \lambda', \mu' \rangle \iff (\lambda \geq_L \lambda' \text{ and } \mu \leq_L \mu').$$

Indeed, the decision maker naturally prefers higher possibility of getting the best outcome and lower possibility of getting the worst one. As the order on scale L_2 is total, L_2 is in fact monodimensional.

Here, the basic utility assignment which takes values in binary scale L_2 is denoted by $u : X \to L_2$. Operator $\vee$ is extended as an operator on $L_2 \times L_2$:

$$\langle \lambda, \mu \rangle \vee \langle \lambda', \mu' \rangle = \langle \lambda \vee \lambda', \mu' \vee \mu' \rangle.$$

Note this operator is not operator max on L_2 (computed as $\max(\langle \lambda, \mu \rangle, \langle \lambda', \mu' \rangle) = \langle \lambda \vee \lambda', \mu \wedge \mu' \rangle$). Operator $\wedge$ is extended as an operator on $L \times L_2$:

$$\lambda' \wedge \langle \lambda, \mu \rangle = \langle \lambda' \wedge \lambda, \lambda' \wedge \mu \rangle.$$

In the same way as operator max, operator min on L_2 is given by $\min(\langle \lambda, \mu \rangle, \langle \lambda', \mu' \rangle) = \langle \lambda \wedge \lambda', \mu \vee \mu' \rangle$. PU is defined as a utility function from $\mathcal{F}$ to L_2:

$$PU(f) = \bigvee_{s \in S} \big(\pi(s) \wedge u(f(s))\big). \tag{1}$$

As noted by [8], when the basic utility has the following restricted form: $\forall x \in X, \exists \lambda \in L, u(x) = \langle \lambda, 1_L \rangle$ then PU is an optimistic utility. Dually when we have $\forall x \in X, \exists \mu \in L, u(x) = \langle 1_L, \mu \rangle$, PU is a pessimistic utility. These remarks imply that optimistic and pessimistic utilities exploit only half of scale L_2.

PU is said to be *autodual* as it is its own dual. This property is shared by EU. Intuitively, it means that the decision model when stating a preference between two acts looks at how well and how bad these two acts do.

[1] This particular framework should not be confused with the bipolar one as stated by [14] for example. Here only one possibility distribution is used while in the bipolar setting two possibility distributions are exploited.

Sugeno Integral. We now present the Sugeno integral, which can be thought of as a very general decision criterion where uncertainty is assumed to be modeled by a capacity. It can be seen as the qualitative counterpart of Choquet Expected Utility (CEU) [15, 16], which is a generalization of EU. CEU has a higher descriptive power than that of EU. Being qualitative counterparts of EU, optimistic and pessimistic utilities are special cases of Sugeno integrals when the capacity is taken as respectively a possibility and a necessity distributions.

The utility of an act described by a Sugeno integral writes:

$$SU(f) = \bigvee_{x \in X} \big(\mu(x) \wedge \sigma(F_x)\big) \tag{2}$$

where $F_x = \{s \in S : f(s) \succsim_X x\}$. While the quantitative models (CEU or EU) compute a weighted average, the Sugeno integral $SU(f)$ is the median of the utilities and uncertainty values in the following set [17]: $\{\mu(x) : x \in X\} \cup \{\sigma(F_x) : x \in X, x \succ_X 0_X\}$.

2.3 Axiomatizations of Qualitative Utilities

Sugeno Integral. [11] axiomatized this criterion using the following axioms:

Sav1 Preference relation $\succsim_{\mathcal{F}}$ over $\mathcal{F}$ is a total preorder.
WS3 $x \succsim_X y \Rightarrow \forall A \in E, \forall h \in \mathcal{F}, xAh \succsim_{\mathcal{F}} yAh$
Sav5 $\exists x, x' \in X, x \succ_X x'$
RCD $\forall f, g \in \mathcal{F}, \forall x \in X, g \succ_{\mathcal{F}} f$ and $\mathbf{x} \succ_{\mathcal{F}} f \Rightarrow g \wedge \mathbf{x} \succ_{\mathcal{F}} f$
RDD $\forall f, g \in \mathcal{F}, \forall x \in X, f \succ_{\mathcal{F}} g$ and $f \succ_{\mathcal{F}} \mathbf{x} \Rightarrow f \succ_{\mathcal{F}} g \vee \mathbf{x}$

Axiom $Sav1$ states that any two acts can be compared and the preference relation is transitive. Axiom $WS3$ declares that when there is a preference between two consequences, this preference can not reverse if considering two acts giving these consequences on a certain event and the same consequences on the complementary event. Axiom $Sav5$ is enforced in order to avoid triviality. Axiom RCD states that lowering the consequences of an act f, which is preferred to an act g, to a constant consequence that is also preferred to g still yields a better act than g. Finally axiom RDD is the dual of RCD. It says that an act f is preferred to an act g even if the worst consequences of g are improved to a constant consequence that is less preferred than f. As a side note, EU verifies neither RCD nor RDD [11]. Sugeno integrals have been characterized by [11]:

Theorem 1 *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $Sav5$, RCD and RDD then there are a qualitative scale $(L, \geq_L)$, a basic utility assignment $\mu : X \to L$ and a capacity $\sigma : E \to L$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow SU(f) \geq_L SU(f')$.*

Optimistic and Pessimistic Utilities. To enforce the optimistic utility, axiom RDD needs to be replaced by a stronger axiom:

OPT $\forall f, g \in \mathcal{F}, \forall A \in E, f \succ_{\mathcal{F}} fAg \Rightarrow gAf \succsim_{\mathcal{F}} f$

This axiom says that if an act can be strictly downgraded in a certain event, this event is judged plausible enough to attract all the attention of the decision maker and what happens on the complementary event is not relevant.

The representation theorem for the optimistic utility as stated by [11]:

Theorem 2 *If $\succsim_\mathcal{F}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $Sav5$, RCD and OPT then there are a qualitative scale $(L, \geq_L)$, a basic utility assignment $v : X \to L$ and a possibility distribution $\pi : E \to L$ s.t. $f \succsim_\mathcal{F} f' \Leftrightarrow U^+(f) \geq_L U^+(f')$.*

For the pessimistic utility, RCD is to be replaced by:

PES $\forall f, g \in \mathcal{F}, \forall A \in E, fAg \succ_\mathcal{F} f \Rightarrow f \succsim_\mathcal{F} gAf$

This axiom, which is the dual of OPT states that if an act is strictly improved in a certain event then it can not be strictly improved in the complementary event. And the pessimistic utility [11] is characterized by:

Theorem 3 *If $\succsim_\mathcal{F}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $Sav5$, RDD and PES then there are a qualitative scale $(L, \geq_L)$, a basic utility assignment $v : X \to L$ and a possibility distribution $\pi : E \to L$ s. t. $f \succsim_\mathcal{F} f' \Leftrightarrow U^-(f) \geq_L U^-(f')$.*

Binary Possibilistic Utility. PU is a Sugeno integral [12] with uncertainty modeled by *autodual possibilistic* measures, i.e., couples $\langle \pi(A), \pi(\overline{A}) \rangle$.

For any act $h \in \mathcal{F}$, $h^{\prec_\mathcal{F}}$ denotes the set of acts $\{f \in \mathcal{F} : f \prec_\mathcal{F} h\}$ and $h^{\succsim_\mathcal{F}} = \{f \in \mathcal{F} : f \succsim_\mathcal{F} h\}$. They are respectively the set of acts that are less preferred than h and its complement. The following axioms are required to state our theorem:

AD Preference relation $\succsim_\mathcal{F}$ over $\mathcal{F}$ is autodual
OPT' $\exists h \in \mathcal{F}$, $\mathbf{1_X} \succ_\mathcal{F} h \succ_S \mathbf{0_X}$ s.t. $\forall f, g \in h^{\succsim_\mathcal{F}}, \forall A \in E, f \succ_\mathcal{F} fAg \Rightarrow gAf \succsim_\mathcal{F} f$

Axiom AD states that the preferences of the decision-maker is autodual. Axiom OPT' states that the set of acts can be divided into two parts, where the set of less preferred acts satisfies axiom OPT.

Theorem 4 *If $\succsim_\mathcal{F}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $Sav5$, RCD, OPT' and AD then there are a qualitative scale $(L, \geq_L)$, a basic utility assignment $u : X \to L_2$ and a possibilistic distribution $\pi : E \to L$ such that $f \succsim_\mathcal{F} f' \Leftrightarrow PU(f) \geq_{L_2} PU(f')$.*

2.4 Semi-Qualitative Utilities

Binary Likelihood-Based Utility. Consider the statistical problem described by $(Y, \Theta, P_{\theta \in \Theta})$ where Y is a random variable whose probability distribution is unknown, Θ is a parameter set and $P_{\theta \in \Theta}$ is a family of probability distributions. The real probability distribution of Y is assumed to be one of $P_{\theta \in \Theta}$. When observing a new data $Y = y$, one can compute the likelihood of a parameter θ by $P_\theta(Y = y)$. Thus one can define a likelihood function valued on $L = [0, 1]$

by $l(\theta) = P_\theta(Y = y)$. As noted by [9], likelihood can be viewed as a possibility measure. So from now on, we will denote likelihood l simply by π.

When using a likelihood function as a uncertainty representation, [9] justified axiomatically binary likelihood-based utility (LU) in a vNM setting.

LU is then defined as a utility function from $\mathcal{F}$ to L_2:
$LU(f) = \bigvee_{s\in S} \big(\pi(s)u(f(s))\big)$ where $u : X \to L_2$ is a basic utility assignment.

This criterion can be considered as an intermediate model between qualitative model such as PU and quantitative model such as EU. To the best of our knowledge, it has not yet received a Savagean axiomatization. One of the aims of this paper is to fill this gap.

Spohn's Disbelief Function-Based Utility. A Spohn's disbelief function that is a uncertainty representation is defined by a set function $\kappa : E \to \mathbb{N} \cup \{\infty\}$ satisfying $\kappa(S) = 0$, for event $A \neq \emptyset$, $\kappa(A) = \min_{s\in A} \kappa(\{s\})$ and $\kappa(\emptyset) = \infty$. The value of κ measures the surprise of the decision-maker.

Here scale $(L, \geq_L)$ can be defined by $(\mathbb{N} \cup \{\infty\}, \leq)$. Note that as the order is reversed, $\vee$ and $\wedge$ are exchanged. Spohn's disbelief function-based utility (κU) has been axiomatically justified in a vNM setting by [10]. κU is then defined as a utility function from $\mathcal{F}$ to L_2:

$$\kappa U(f) = \bigvee_{s\in S} \big(\kappa(s) + u(f(s))\big). \tag{3}$$

where $u : X \to L_2$ is a basic utility assignment.

This criterion is related to other decision models. It can be obtained from EU when using orders of magnitude of probabilities and utilities [10]. Besides it is isomorphic to a binary likelihood-based utility when equivalence classes of consequences and equivalence classes of events are countable.

2.5 Generalized Qualitative Utility

[7] have axiomatized, in a vNM setting, generalized qualitative utility (GU), which is a class of decision models that use a possibilistic uncertainty representation. GU generalizes optimistic and pessimistic qualitative utilities.

Before presenting GU, let us recall the definition of a it triangular norm or *t-norm*, denoted $\top$, which are operators defined on $[0, 1] \times [0, 1]$ into $[0, 1]$. Operator $\top$ is a t-norm iff for all $x, y, z \in [0, 1]$:

T1 Commutativity: $x\top y = y\top x$
T2 Associativity: $x\top(y\top z)) = (x\top y)\top z$
T3 Monotony: $x\top y \geq x\top z$ if $y \geq z$
T4 Limit condition: $x\top 1 = x$

min, $\times$ are two classic examples of t-norms. For a given t-norm $\top$, one can define a *triangular conorm* or *t-conorm*, denoted $\perp$, which are operators defined on $[0, 1] \times [0, 1]$ into $[0, 1]$ by $x\perp y = 1 - \big((1 - x)\top(1 - y)\big)$. Thus, operator $\perp$ is a t-conorm iff it satisfies $T1$ to $T3$ and for all $x \in [0, 1]$:

S4 Limit condition: $x \bot 0 = x$

Two examples of t-conorms are max and probabilistic sum defined by $x \bot y = x + y - xy$. GU takes its value on the same scale as the possibility scale and thus has two versions: generalized optimistic and generalized pessimistic utilities, which are functions from $\mathcal{F}$ to $[0,1]$. Generalized optimistic utility GU^+ and generalized pessimistic utility GU^- write for any t-norm $\top$ and its associated t-conorm $\bot$:

$$GU^+(f) = \bigvee_{s \in S} \big(\pi(s) \top v(f(s))\big) \qquad GU^-(f) = \bigwedge_{s \in S} \big(n(\pi(s)) \bot v(f(s))\big)$$

It is easy to see that those criteria are generalizations of U^+ and U^- when $\top = \wedge$ and generalizations of LU^+ and LU^- when $\top = \times$.

While optimistic/pessimistic utilities (and binary possibilistic utility) are special cases of Sugeno integrals, the last two criteria GU^+ and GU^- can be viewed as instances of a generalized Sugeno integral that we introduce now.

3 Proposed Axiomatizations

Generalized Sugeno Integrals. A utility function defined as a generalized Sugeno integral (GSU) is a function from $\mathcal{F}$ to L:

$$GSU(f) = \bigvee_{x \in X} \big(\mu(x) \top \sigma(F_x)\big) \tag{4}$$

where $\top$ is a t-norm[2] on L, $F_x = \{s \in S : f(s) \succsim_X x\}$ and $\sigma : E \to L$ is a monotone set function representing the belief of the decision-maker. Note that we do not define GSU as a function taking values in $[0,1]$ as we want to be general and because it is needed afterwards. Thus GSU could be defined on various scales, L_2 for example.

We now present the axiom that we will use in the axiomatization of GSU.

EX $\big(xA0_X \sim_{\mathcal{F}} 1_X B 0_X$ and $yA0_X \sim_{\mathcal{F}} 1_X C 0_X\big) \Rightarrow xC0_X \sim_{\mathcal{F}} yB0_X$

Assume that $x \succsim_X y$. We say that binary act $1_X B 0_X$ is preferred to $1_X C 0_X$ with a premium defined by the couple (x, y). This axiom states that if a binary act is preferred to another one with a certain premium (x, y), this preference can be cancelled by switching the two consequences (x, y).

We also use axiom $WS4$ introduced by [18].

WS4 $xAx' \succsim xBx' \Rightarrow yAy' \succsim yBy'$

where $x \geq_P y >_P y' >_P x', A, B \subseteq S$. This axiom entails some coherence in the preferences between binary acts. Changing to less extreme consequences in equivalent binary acts keep those acts equivalent. This axiom is not needed in the previous axiomatizations as it is implied by the set of axioms used in the representation theorems. However, here as we work in a relaxed framework, this axiom has to be specified.

Thanks to these axioms, we can characterized generalized Sugeno integrals:

[2] To simplify the exposition, we call an operator on L a t-norm iff it satisfies $T1$ to $T4$ expressed on L. The same convention will be taken for t-conorm.

Theorem 5 *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX and RDD then there are a totally ordered scale L equipped with a t-norm $\top$, a basic utility assignment $\mu : X \to L$ and a capacity $\sigma : E \to L$ such that $f \succsim_{\mathcal{F}} f'$ $\Leftrightarrow GSU(f) \geq_L GSU(f')$.*

Proof. The proof is quite similar to that of the representation theorem of Sugeno integrals [11] except for their steps 3 et 4 as we do not assume here axiom RCD. We outlined the proof, which follows in four steps:

Step 1. The equivalence classes of $\succsim_{\mathcal{F}}$ define the utility scale L that is completely ordered by $Sav1$. We define $GSU(f)$ as the equivalence class of f and $\mu(x)$ as that of x by restricting $\succsim_{\mathcal{F}}$ on X.

Step 2. By $WS3$ and $Sav5$, we can define $\sigma(A)$ as a monotone set function $E \to L$ with $GSU(1_X A 0_X)$.

Step 3. On binary acts, $GSU(xA0_X)$ can be expressed as a function of $\mu(x)$ and $\sigma(A)$ as $x \sim_X x'$ and $A \sim_S A'$ imply $xA0_X \sim_{\mathcal{F}} x'A'0_X$ by $WS3$ and $WS4$. Define function $T : \mu(X) \times \sigma(E) \to L$ by $GSU(xA0_X) = T(\mu(x), \sigma(A))$. Obviously T is monotone. Let $L' = \mu(X) \cap \sigma(E) \subseteq L$. Function T is commutative on $L' \times L'$ by EX. Indeed letting $A = S$, we have $T(\mu(x), \sigma(C)) = T(\mu(y), \sigma(B))$, $\sigma(B) = \mu(x)$ and $\sigma(C) = \mu(y)$. Using commutativity on $L' \times L'$ and by EX again, T is associative on $L' \times L'$. By definition of GSU and μ, $\forall a \in L, T(a, 1_L) = a$. Finally, T can extended on L as an operator satisfying $T1$ to $T4$. As T is a t-norm, from now on, we will use the infix notation. We have $GSU(xA0_X) = \mu(x) \top \sigma(A)$.

Step 4. Note any act f can be written in the following form $y_1 A_1 y_2 A_2 \ldots y_k A_k 0_X$ where $k < n$ is the number of different non null consequences of the act, $y_1 \succ_X \ldots \succ_X y_k$ and $A_1 \subseteq A_2 \subseteq \ldots A_n$.
Let us prove by induction over k that it exists $i = 1, \ldots, k$, $f \sim_{\mathcal{F}} y_i A_i 0_X$. For $k = 1$, it is true as $f = y_1 S 0_X$. For a $k > 0$, assume that the proposition is true for any $j <= k$. Let $f = y_1 A_1 y_2 A_2 \ldots y_{k+1} A_{k+1} 0_X$ be an act in $\mathcal{F}$.
The following fact can be proved: $\forall i = 1, \ldots, k{+}1$, $\mathbf{y_i} \succsim_{\mathcal{F}} f$ implies $fA_i0_X \succsim_{\mathcal{F}} f$ or $\mathbf{y_{i-1}} \succsim_{\mathcal{F}} f$. Indeed, if $\mathbf{y_i} \succsim_{\mathcal{F}} f$, $fA_i0_X \prec_{\mathcal{F}} f$ and $\mathbf{y_{i-1}} \prec_{\mathcal{F}} f$, then by RCD, $f \succ_{\mathcal{F}} fA_i y_{i-1}$, which contradicts with the monotonicity of $\succsim_{\mathcal{F}}$ implied by $Sav1$ and $WS3$ (Lemma 4 of [11]). Then there is an $i = 1, \ldots, k+1$ such that $fA_i0_X \succsim_{\mathcal{F}} f$. By monotonicity again, we have $f \succsim_{\mathcal{F}} fA_i w_X$. Consequently $f \sim_{\mathcal{F}} fA_i w_X$. The induction assumption applied on $fA_i w_X$ yields the result.
Finally, $GSU(f) = \bigvee_{i=1,\ldots,n} \mu(x_i) \top \sigma(A_i)$.

∎

Generalized Qualitative Utility. We can now axiomatize GU^+ as it is a generalized Sugeno integral when using a possibility distribution by enforcing OPT instead of RDD. To better understand OPT, we prove a result that shows how strong it is as it imposes that only one pair of consequence and event is important in determining the value of an act.

Proposition 1 *If $\succsim_{\mathcal{F}}$ satisfies $Sav1$, $WS3$ and OPT then for any act $f \in \mathcal{F}$, which can be written $f = x_1 A_1 x_2 A_2 \ldots x_n A_n$, $f \sim_{\mathcal{F}} \max_{i=1,\ldots,n} x_i A_i 0_X$.*

Proof. Similar to the induction used in step 4 of the proof of Theorem 5. ∎

The representation theorems for the optimistic and the pessimistic cases can be written as follows:

Theorem 6 *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX and OPT then there are a basic utility assignment $\mu : X \to [0,1]$ and a possibility distribution $\pi : E \to [0,1]$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow GU^+(f) \geq GU^+(f')$.*

If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX and PES then there are a basic utility assignment $\mu : X \to [0,1]$ and a possibility distribution $\pi : E \to [0,1]$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow GU^-(f) \geq GU^-(f')$.

Proof. Interval $[0,1]$ can be used as a scale as E is a sigma-algebra and X is finite. The result for GU^+ is implied by Theorem 5 and by axiom OPT that entails that σ is a possibility distribution. The pessimistic version is obtained by duality. ∎

To obtain instances of the class of generalized qualitative utilities, one needs to enforce more restrictive axioms. Doing so, the differences of two instances can be highlighted by comparing their two sets of axioms. In the following section, we propose extra axioms to be added to get likelihood-based utilities.

Optimistic/Pessimistic Likelihood-Based Utilities. Likelihood-based utility as presented in section 2.4 is a binary utility. One can restrict it to its optimistic and pessimistic versions. If basic utility assignment u only takes values in $\{\langle \lambda, 1_L \rangle : \lambda \in L\}$ then LU can be said to be an optimistic utility. Dually when u only takes values in $\{\langle 1_L, \mu \rangle : \mu \in L\}$ then LU can be qualified pessimistic. The two thus defined criteria would be counterparts of optimistic and pessimistic qualitative utilities (Section 2.2).

For a generalized optimistic qualitative utility to be an optimistic likelihood-based utility, one has to enforce this extra axiom that implies that the t-norm used in GU^+ is a strict t-norm:

ST $A \succ_S A' \Rightarrow \forall x \neq 0_X, xA0_X \succ_{\mathcal{F}} xA'0_X$
C1 $\forall n \in \mathbb{N}, xA_n 0_X \succsim_{\mathcal{F}} f \Rightarrow x(\lim_{n\to\infty} A)0_X \succsim_{\mathcal{F}} f$
C2 $\forall n \in \mathbb{N}, x_n A 0_X \succsim_{\mathcal{F}} f \Rightarrow (\lim_{n\to\infty} x_n)A0_X \succsim_{\mathcal{F}} f$

Axiom ST has a natural interpretation: Binary acts that gives with stronger belief the better outcome are strictly preferred. Axioms $C1$ and $C2$ are two continuity conditions. The representation theorems for the optimistic and pessimistic cases can then be stated as follows:

Theorem 7 *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX, OPT, ST, $C1$ and $C2$ then there are a basic utility assignment $\mu : X \to [0,1]$ and a possibility distribution $\pi : E \to [0,1]$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow LU^+(f) \geq LU^+(f')$.*

If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX, PES, ST, $C1$ and $C2$ then there are a basic utility assignment $\mu : X \to [0,1]$ and a possibility distribution $\pi : E \to [0,1]$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow LU^-(f) \geq LU^-(f')$.

Proof. Theorem 6 yields a generalized optimistic utility GU defined with some t-norm. Axioms ST, $C1$ and $C2$ then entail that we can build a strict t-norm. Consequently GU can be transformed into LU^+ using the multiplicative generator of the t-norm [19]. ∎

4 Extension to Binary Utilities

Generalized Binary Qualitative Utility. Generalized optimistic/pessimistic qualitative utilities can be easily unified in the same fashion as optimistic and pessimistic utilities with binary possibilistic utility. We now present this quick development and call the unifying criterion the generalized binary qualitative utility (GPU).

In this section, scale L is taken as $[0,1]$ on which a t-norm $\top$ is defined. T-norm $\top$ is extended as an operator on $L \times L_2$: $\lambda' \top \langle \lambda, \mu \rangle = \langle \lambda' \top \lambda, \lambda' \top \mu \rangle$.

GPU is then defined as a function from $\mathcal{F}$ to L_2:

$$GPU(f) = \bigvee_{s \in S} \big(\pi(s) \top u(f(s)) \big). \tag{5}$$

When the basic utility assignment has the following restricted form: $\forall x \in X, \exists \lambda \in L, u(x) = \langle \lambda, 1_L \rangle$ then GPU becomes GU^+. Dually when we have $\forall x \in X, \exists \mu \in L, u(x) = \langle 1_L, \mu \rangle$, GPU becomes GU^-.

[20] has proposed an axiomatization for GPU in a vNM setting as GPU is in fact an instance of algebraic expected utility. But no Savagean axiomatization for this criterion has been proposed yet. We now propose such an axiomatization:

Theorem 8 *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX, OPT' and AD then there are a basic utility assignment $\mu : X \to L_2$ and a possibility distribution $\pi : E \to L$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow GPU(f) \geq_L GPU(f')$.*

Proof. For any operator $\top$ on L, operator $\top_2$ can be defined on L_2 by $\langle \lambda, \mu \rangle \top_2 \langle \lambda', \mu' \rangle = \langle \lambda \top \lambda', \mu \bot \mu' \rangle$ where $\bot$ is the associated t-conorm of $\top$.

It is easy to verify that if $\top$ is a t-norm on L then $\top_2$ is a t-norm on L_2. Using this fact, the same proof as in Prop. 2 of [12] can show that GPU is a GSU defined on L_2 with t-norm $\top_2$ and using autodual possibilistic measures as uncertainty representation. The result is then implied by Theorem 5 and by axioms OPT' and AD entailing that σ is autodual possibilistic. ∎

Binary Likelihood-based Utility. Finally with the help of the previous results, a representation theorem is stated for binary likelihood-based utility.

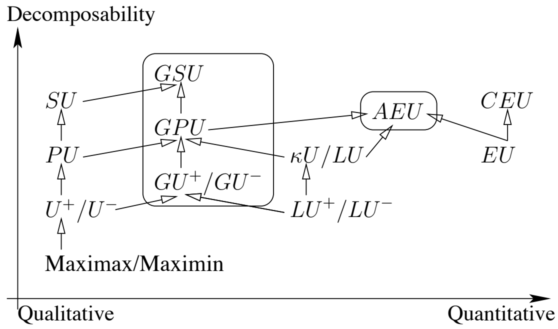

Fig. 1. Generalized Qualitative Utilities.

Theorem 9 *If $\succsim_{\mathcal{F}}$, a preorder over $\mathcal{F}$, satisfies $Sav1$, $WS3$, $WS4$, $Sav5$, EX, OPT', ST, $C1$, $C2$ and AD then there are a basic utility assignment $\mu : X \to L_2$ and a possibility distribution $\pi : E \to L$ such that $f \succsim_{\mathcal{F}} f' \Leftrightarrow LU(f) \geq_L LU(f')$.*

As a side note, one can check easily that Spohn's disbelief function-based utility satisfies all the axioms and in the construction of LU^+ and LU^-, one can use the additive generator of the t-norm [19] to build that criterion. However one extra technical axiom needs to be added to ensure that the scale is countable.

5 Conclusion

In this paper, we have proposed axiomatizations in a Savage framework for previous proposed decision models that are instances of generalized qualitative models. Doing so, a better picture of how those criteria relate to each other can be drawn (Fig. 1). Models represented in a frame are general classes of decision models. An arrow can be read as "is an instance of". Except for those general models, decision criteria on the left hand-side are more qualitative in the sense that they require less information to be implemented whereas as we move on the right, models become more quantitative. Models on the bottom of the figure use a uncertainty representation that is decomposable while on the top, we have more general models that relax the hypothesis of decomposability.

Our work reveals the common properties that all those decision models share and in the same time underlines their differences from a decision-theoretic point of view. Besides the proposed axioms can be used to test if a decision-maker uses such qualitative utilities as decision criteria and represents his beliefs with possibility measures, likelihood functions or Spohn's disbelief functions. Finally as a by-product, during the process of axiomatizing those utilities, we introduced a generalization of Sugeno Integrals.

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