Two Factor Additive Conjoint Measurement With One Solvable Component

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* I am greatly indebted to Jean-Yves Jaffray, Peter Fishburn and Peter Wakker for their very useful comments and suggestions.
Abstract

This paper addresses conditions for the existence of additive separable utilities. It considers mainly two-dimensional Cartesian products in which restricted solvability holds w.r.t. one component, but some results are extended to n-dimensional spaces. The main result shows that, in general, cancellation axioms of any order are required to ensure additive representability. More precisely, a generic family of counterexamples is provided, proving that the \((m + 1)\)st-order cancellation axiom cannot be derived from the \(m\)th order cancellation axiom when \(m\) is even. However, a special case is considered, in which the existence of additive representations can be derived from the independence axiom alone. Unlike the classical representation theorems, these representations are not unique up to strictly positive affine transformations, but follow Fishburn’s (1981) uniqueness property.
1 Introduction

Additive conjoint measurement has been extensively studied since 1960 and Debreu’s celebrated paper. For instance, Jaffray (1974a, 1974b) give necessary and sufficient conditions ensuring the existence of additive representations, which are closely related to cancellation axioms (see Krantz et al. (1971, p.340–344)). Other necessary and sufficient conditions can be found in Krantz et al. (1971, p.430) and in Fishburn (1992).

However, the lack of testability of high order cancellation axioms led researchers to look for other axioms that were nonnecessary but easier to test. Thus, Luce and Tukey (1964), Krantz (1964), Luce (1966), Fishburn (1970, p.58) and Krantz et al. (1971, p.257) show sufficient conditions of an algebraic nature, as well as uniqueness properties of the additive representations. Debreu (1960) and Wakker (1989, p.49) give their counterpart in a topological framework. Fuhrken and Richter (1991) gave necessary and sufficient conditions for the existence of continuous additive representations on connected topological spaces. All these papers assume that some structural nonnecessary conditions — called unrestricted or restricted solvability in the former approach and connectedness of the topological spaces in the latter — hold w.r.t. every component of the Cartesian product. But these assumptions are particularly restrictive because they simply do not hold in many practical situations. For instance, in economics, the first component may describe a waiting time and the second one the object waited for (in this case, using an exponential transform, one gets a multiplicative model). In medical decision making, the first component may refer to the duration of a treatment, and the second one to the level of quality of life induced by it. Another example can be found in Gonzales and Jaffray (1997), in which certainty effects make a restricted solvability assumption unrealistic.

Therefore, it is of interest to study Cartesian products in which solvability or connectedness does not hold w.r.t. every component. Gonzales (1996a, 1996b) describes testable conditions ensuring additive representability when unrestricted or restricted solvability holds w.r.t. at least two components. But, to my knowledge, the only paper dealing with spaces in which restricted solvability holds w.r.t. exactly one component is Fishburn (1981), which studies the uniqueness property of additive utilities. In this paper, no conditions are mentioned ensuring the existence of such representations. Those are addressed in the present paper. The choice of using solvability rather than connectedness was recommended by Wakker (1988).

In section 2, axioms needed in the paper are given. Section 3 presents a generic family of counterexamples showing that, in general, cancellation axioms of any order are needed to entail additive representability on two-dimensional Cartesian products in which unrestricted solvability holds w.r.t. only one component. This can be seen as a complement to Fishburn (1997a, 1997b, 1997c), which gives the same result for finite sets, and as an extension of Scott and Suppes (1958). The result is trivially extended to n-dimensional Cartesian products in which only one component satisfies unrestricted solvability. Section 3 also studies the relationship between cancellation axioms of different orders. In particular, it is proved that the second order cancellation axiom is equivalent to the conjunction of independence and the Thomsen condition; this result was already well known when restricted solvability holds w.r.t. all the components (see e.g., Krantz et al. (1971, p.257)) or when unrestricted solvability holds w.r.t. at
least two components (see Gonzales (1996a)) or even when a component is finite and
the other one is binary (see Krantz et al. (1971, p.452)).

Section 4 describes how a “weak” structural assumption w.r.t. the second component,
namely equal spacedness, combined with restricted solvability w.r.t. the first component,
independence and an Archimedean axiom, implies additive representability. This is an

All proofs are given in the appendix.

2 The axiomatic system

Consider a Cartesian product $X = X_1 \times X_2$ and a weak order $\succsim$ over $X$, i.e., $\succsim$ is
complete (for all $x, y \in X$, $x \succsim y$ or $y \succsim x$) and transitive. Let $x \sim y \equiv [x \succsim y$ and
$y \succsim x]$, $x \succ y \equiv [x \succsim y$ and Not$(y \succsim x)]$, and $x \preceq y \equiv y \succsim x$. $\succsim$ is representable
by an additive utility function $u : X \rightarrow \mathbb{R}$ if there exist mappings $u_1 : X_1 \rightarrow \mathbb{R}$ and
$u_2 : X_2 \rightarrow \mathbb{R}$ such that:

$$u(x_1, x_2) = u_1(x_1) + u_2(x_2) \text{ for all } (x_1, x_2) \in X$$

$$(x_1, x_2) \succsim (y_1, y_2) \iff u(x_1, x_2) \geq u(y_1, y_2) \text{ for all } (x_1, x_2), (y_1, y_2) \in X.$$

The following four axioms are well known to be necessary for additive representability
on any set $X:

Axiom 1 (independence) For all $x, y \in X$, $(x_1, x_2) \succsim (y_1, y_2) \iff (y_1, x_2) \succsim (y_1, y_2)$
and $(x_1, x_2) \succsim (y_1, x_2) \iff (x_1, y_2) \succsim (y_1, y_2)$.

By axiom 1, weak orders $\succsim_1$ and $\succsim_2$, induced by $\succsim$ respectively on $X_1$ and $X_2,
can be defined for all $x_1, y_1 \in X_1$, $x_2, y_2 \in X_2$ as: $x_1 \succsim_1 y_1 \iff$ [there exists $c_2 \in X_2$
such that $(x_1, c_2) \succsim (y_1, c_2)$], and $x_2 \succsim_2 y_2 \iff$ [there exists $c_1 \in X_1$ such that $(c_1, x_2) \succsim (c_1, y_2)$].

Axiom 2 (Thomsen condition) For every $x_1, y_1, z_1 \in X_1$, $x_2, y_2, z_2 \in X_2$, if $(x_1, z_2)$
$\sim (z_1, y_2)$ and $(z_1, x_2) \sim (y_1, z_2)$, then $(y_1, y_2) \sim (x_1, x_2)$.

The Thomsen condition can be generalized by substituting indifference relations $\sim$
by weak preference relations $\succeq$, thus resulting in the following axiom:

Axiom 3 (Generalized Thomsen condition) For every $x_1, y_1, z_1 \in X_1$, $x_2, y_2, z_2 \in
X_2$, if $(x_1, z_2) \succeq (z_1, y_2)$ and $(z_1, x_2) \succeq (y_1, z_2)$, then $(y_1, y_2) \succeq (x_1, x_2)$.

Axiom 4 ((C2) : second order cancellation axiom) Consider $(x_1^i, x_2^i)$ and $(y_1^i, y_2^i),
i \in \{1, 2, 3\}$, such that $(x_1^i, x_2^i, x_3^i)$ is a permutation of $(y_1^j, y_2^j, y_3^j)$ for $j = 1, 2$. If
$(x_1^1, x_2^1) \succeq (y_1^1, y_2^1)$ and $(x_1^2, x_2^2) \succeq (y_1^2, y_2^2)$, then $(x_1^3, x_2^3) \succeq (y_1^3, y_2^3)$.

The following two structural conditions are usually assumed w.r.t. every component
of $X$, but, in what follows, they will only be w.r.t. one component.

Axiom 5 (unrestricted solvability w.r.t. the first component) For every $y \in X$
and every $x_2 \in X_2$, there exists $x_1 \in X_1$ such that $y \sim (x_1, x_2)$.
Axiom 6 (restricted solvability w.r.t. the first component) For every \( y \in X \), if \((a_1, x_2) \preceq y \preceq (b_1, x_2)\), then there exists \( x_1 \in X_1 \) such that \( y \sim (x_1, x_2) \).

Clearly, when unrestricted solvability holds, then restricted solvability holds as well, so that representation theorems involving restricted solvability are more general than the ones involving unrestricted solvability. These two axioms are structural conditions because: i) they are nonnecessary for additive representability; and ii) they strengthen axioms 1 through 4 to the point that they induce additive representability.

Throughout, when \( f \) and \( g \) are functions, \( f \circ g(\cdot) = f(g(\cdot)) \); if, moreover, \( p \) is a strictly positive integer, \( f^p = f \circ f \circ \cdots \circ f \) (\( p \) times), and \( f^{-1} \) denotes the inverse function of \( f \), i.e., \( f \circ f^{-1} \) is the identity function.

3 Counterexamples

3.1 Independence and the Thomsen conditions

Lemma 1 Let \( \succsim \) be a weak order on a two-dimensional Cartesian product \( X \). Then

\[
\left( \begin{array}{c}
\text{the independence axiom (axiom 1)} \\
\text{the generalized Thomsen condition (axiom 3)}
\end{array} \right) \iff \text{the second order cancellation axiom (C}_2) \nonumber
\]

This lemma is rather general since, unlike Krantz et al. (1971), it requires no structural condition such as finiteness of \( X_i \) or solvability w.r.t. \( X_1 \). However, when the latter holds, the generalized Thomsen condition can be substituted by the classical Thomsen condition, hence the following lemma:

Lemma 2 Let \( \succsim \) be a weak order on a two-dimensional Cartesian product \( X \) satisfying unrestricted solvability w.r.t. the first component (axiom 5). Then

\[
\left( \begin{array}{c}
\text{the independence axiom (axiom 1)} \\
\text{the Thomsen condition (axiom 2)}
\end{array} \right) \iff \text{the second order cancellation axiom (C}_2) \nonumber
\]

This lemma is a slight generalization of property P1* \( \Rightarrow \) P1 of theorem 5.3 in Fishburn (1970) (because, in lemma 2, solvability is not required w.r.t. the second component). However, the proof I present here is completely different from the one written by Fishburn.

It is a classical result that in two-dimensional Cartesian products, the second order cancellation axiom is not implied by independence alone: one must add the Thomsen condition to get this implication. A well known counterexample (see e.g., Wakker (1989, p.71)\(^1\)) is the following one: \( X = \mathbb{R} \times \mathbb{R} \) and \( \succsim \) is defined by the utility function \( U(x_1, x_2) = x_1 + x_2 + \min\{x_1, x_2\} \).

\(^1\)In this example, the Thomsen condition is substituted by the hexagon condition, which is somewhat less restrictive. But according to Karni and Safra (1997), under restricted solvability w.r.t. both components and independence, the Thomsen condition is equivalent to the hexagon condition.
3.2 Relations between different cancellation axioms

In this subsection, a new condition, called a symmetric cancellation axiom and denoted \((S_{m+1})\), is introduced. This is a particular symmetric case of the classical cancellation axiom \((C_{m+1})\) (defined below). As we shall see, it fills the gap between \((C_m)\) and \((C_{m+1})\) when unrestricted solvability holds w.r.t. one component of \(X\).

**Axiom 7 \(((C_m) : mth order cancellation axiom)\)** Let \(m \geq 2\). Consider \(m+1\) elements \((x_1^i, x_2^i)\) of \(X\), \(i = 1, \ldots, m+1\), and \(m+1\) elements \((y_1^j, y_2^j)\) such that \((y_1^1, \ldots, y_1^{m+1})\) and \((y_2^1, \ldots, y_2^{m+1})\) are permutations of \((x_1^1, \ldots, x_1^{m+1})\) and \((x_2^1, \ldots, x_2^{m+1})\) respectively. If \((x_1^i, x_2^i) \succsim (y_1^j, y_2^j)\) for all \(i = 1, 2, \ldots, m\), then \((x_1^{m+1}, x_2^{m+1}) \succsim (y_1^{m+1}, y_2^{m+1})\).

Note that the order of the cancellation axiom follows the notation of Krantz et al. (1971, p.343) (Fishburn (1970) and Jaffray (1974a) would have called it the \((m+1)\)st order cancellation axiom).

**Axiom 8 \(((S_{m+1}) : (m+1)st order symmetric cancellation axiom)\)** Suppose that \(m\) is an even number. Let \(k = (m+2)/2\). Consider elements \(x_1^i, x_2^i \in X_1\), \(x_3^i, \ldots, x_k^i \in X_2\) and \(y_1^j, \ldots, y_k^j \in X_2\) such that, for all \(i \leq k\) and \(j > k\), \(x_1^i \neq x_1^j\) and for all \(i, j < k\), \(x_2^i \neq y_2^j\). Then:

\[
\begin{pmatrix}
(x_1^1, x_2^1) & \succsim & (x_1^k, y_1^j) \\
(x_2^1, x_2^2) & \succsim & (x_2^1, y_2^j) \\
x_2^3, x_2^4 & \succsim & (x_1^1, y_2^j) \\
& \vdots & \\
(x_1^{m+1}, x_2^k) & \succsim & (x_1^{m+1}, y_2^k) \\
\end{pmatrix}
\]

\[\Rightarrow (x_1^{m+2}, y_2^k) \succsim (x_1^{m+1}, x_2^k).\]

Note the symmetry between the left hand side and the right hand side of preference relations \(\succsim\), and between the elements above the horizontal line and those under the line. Note also that when the \(\succsim\)'s are substituted by \(\sim\)'s, the third order symmetric cancellation axiom corresponds to the hexagon condition (see Wakker (1989, p.47–48)).

**Theorem 1** Let \(\succsim\) be a weak order on \(X = X_1 \times X_2\). Assume that unrestricted solvability w.r.t. the first component (axiom 5) holds. Then, for any \(m \geq 2\),

\[
\begin{cases}
\text{if } m \text{ is odd, } & \text{then } (C_m) \iff (C_{m+1}) \\
\text{if } m \text{ is even, } & \text{then } (C_m) + (S_{m+1}) \iff (C_{m+1}).
\end{cases}
\]

3.3 Second and third order cancellation axioms

In this subsection, an example is given which shows that, when unrestricted solvability holds w.r.t. only one component, \((C_2)\) may hold without \((C_3)\) holding. Suppose that \(\succsim\) is represented on \(X = \mathbb{R} \times \{0, 2, 4, 6\}\) by the following utility function:

\[
U(x_1, x_2) = \begin{cases} 
  x_1 + x_2 & \text{if } x_2 \leq 4, \\
  0.5(x_1 \text{ mod } 2)^2 + 2[x_1/2] + 6.5 & \text{if } x_2 = 6.
\end{cases}
\]
Remark that for a fixed value $x_2$ of $X_2$, $U(x, x_2)$ — henceforth denoted $U_{x_2}(x)$ — is a continuous function on $\mathbb{R}$. Hence, working in $X_1 \times \mathbb{R}$, where $\mathbb{R}$ represents the set of values that can be taken by $U(x_1, x_2)$ over $X$, is very attractive because in this space properties of $\succsim$ can be easily visualized (see figure 1).

Now, the problem is to translate the definition of independence and the Thomsen condition into this new space. $(x_1, x_2) \succsim (y_1, x_2) \iff (x_1, y_2) \succsim (y_1, y_2)$ is equivalent to $U_{x_2}(x_1) \geq U_{x_2}(y_1) \iff U_{y_2}(x_1) \geq U_{y_2}(y_1)$. This condition clearly holds since functions $U_x(\cdot)$ are strictly increasing for every $x \in X_2$. $(x_1, x_2) \succsim (y_1, x_2) \iff (y_1, x_2) \succsim (y_1, y_2)$ is equivalent to $U_{x_2}(x_1) \geq U_{y_2}(x_1) \iff U_{x_2}(y_1) \geq U_{y_2}(y_1)$. So, since we consider only continuous functions, it comes to the non intersection of the graph of $U_{x_2}$ by that of $U_{y_2}$. This condition also holds because $0.5(x \text{ mod } 2)^2 + 2|x/2| + 6.5 > x + 4$ for every $x \in \mathbb{R}$. Therefore $\succsim$, as represented by (1), satisfies the independence axiom.

Let us now translate the Thomsen condition. Figure 1 helps in understanding the process: $[(x_1^1, x_2^2) \sim (x_1^1, x_1^2) \text{ and } (x_1^2, x_2^2) \sim (x_1^2, x_2^1)] \Rightarrow (x_1^2, x_2^3) \sim (x_1^3, x_2^2)$. So, in terms of utility functions, $U_{x_2^2}(x_1^2) = U_{x_2^1}(x_2^2)$, $U_{x_2^1}(x_1^1) = U_{x_2^3}(x_2^1)$ and $U_{x_2^3}(x_2^2) = U_{x_2^2}(x_2^3)$. By the first equality, $x_1^2 = U_{x_2^2}^{-1} \circ U_{x_2^2}(x_1^1)$, and by the second equality, $x_1^3 = U_{x_2^1}^{-1} \circ U_{x_2^1}(x_1^1)$. Hence, the third equality can be written as $U_{x_2^3} \circ U_{x_2^1}^{-1} \circ U_{x_2^3}(x_1^1) = U_{x_2^1} \circ U_{x_2^1}^{-1} \circ U_{x_2^1}(x_1^1)$. But this must be true for all $x_1^1$. Hence the Thomsen condition can be written as:

$$
\text{for all } x_1^1, x_2^2, x_3^2 \in X_2 \text{ and all } x \in \mathbb{R}, \quad U_{x_2^1} \circ U_{x_2^1}^{-1} \circ U_{x_2^3}(x) = U_{x_2^1} \circ U_{x_2^1}^{-1} \circ U_{x_2^3}(x).
$$

Now, to show that $\succsim$, as represented by (1), satisfies the Thomsen condition, let me introduce the following lemma:

**Lemma 3** Let $\succsim$ be a weak order on a $X = X_1 \times X_2$. Assume unrestricted solvability w.r.t. the first component (axiom 5) and independence (axiom 1). Suppose that

\[
\text{for every } x_1^1 \preccurlyeq_2 x_2^2 \preccurlyeq_2 x_3^2 \text{ and every } x_1^1, x_1^2, x_1^3 \in X_1, \quad [(x_1^1, x_2^2) \sim (x_1^1, x_1^2) \text{ and } (x_1^2, x_2^3) \sim (x_1^3, x_1^2)] \Rightarrow (x_1^2, x_2^3) \sim (x_1^3, x_2^2). \tag{2}
\]

Then $\succsim$ satisfies the Thomsen condition (axiom 2).
Thus, to prove that $\succeq$ satisfies the Thomsen condition, it is sufficient to prove that:

for all $x_2 \preceq x'_2 \preceq x''_2$ and all $x \in X_1$, $U_{x_2} \circ U^{-1}_{x'_2} \circ U_{x''_2}(x) = U_{x_2} \circ U^{-1}_{x'_2} \circ U_{x''_2}(x)$.  \(3\)

Suppose that $x''_2 \neq 6$. Then (3) is obviously equivalent to $((x + x''_2) - x'_2) + x''_2 = ((x + x''_2) - x'_2) + x''_2$; this relation is clearly satisfied by $U$, as defined by (1). Now suppose that $x''_2 = 6$; then if $x'_2 = 6$, (3) trivially holds, otherwise (3) is equivalent to: $U_6(x + x''_2) = U_6(x) + x''_2 - x'_2$. Since $x''_2 - x'_2$ are multiples of 2, this can be summarized as: $U_6(x + 2) = U_6(x) + 2$. By (1), $U$ obviously satisfies this condition. Hence the Thomsen condition holds for $\succeq$ represented by $U$. By subsection 3.1, we can conclude that the second order cancellation axiom holds.

However, $\succeq$ cannot be represented by an additive utility function because the third order cancellation axiom does not hold. Indeed, $U(1,6) = U(3,4) = 7$, $U(3,0) = U(1,2) = 3$, $U(1,5,2) = U(3,5,0) = 3.5$ and $U(3,5,4) = 7.5 < U(1,5,6) = 7.625$.

### 3.4 Cancellation axioms of order $m + 1$ and $m + 2$

In this subsection we shall see an example proving that the $(m + 1)$st order cancellation axiom does not imply the $(m + 2)$nd order cancellation axiom for $m$ odd when unrestricted solvability holds w.r.t. only one component. Consequently, no cancellation axiom is sufficient to ensure additive representability. Let $\succeq$ be a weak order on $X = \mathbb{R} \times \{0, 2, 4, \ldots, 2m\}$, with $m \geq 3$ odd, represented by the following utility function:

$$U(x_1, x_2) = \begin{cases} \frac{x_1}{2} + x_2 - m^2 - 2m + 2.5 & \text{if } x_2 < 2m, \\ \frac{0.5(x_1 \text{ mod } 2)^2 + 2|x_1|}{2} & \text{if } x_2 = 2m. \end{cases}$$  \(4\)

Then the following lemma holds:

**Lemma 4 (C$_i$)** holds for all $i \in \{0, 1, \ldots, m + 1\}$.

However, (C$_{m+2}$) cannot hold because:

$$\begin{pmatrix}
(2(m - 1) + 1, 0) & \sim & (1, 2(m - 1)) \\
(4(m - 1) + 1, 0) & \sim & (2(m - 1), 1, 2(m - 1)) \\
\vdots & \vdots & \vdots \\
((m + 1)(m - 1) + 1, 0) & \sim & ((m - 1)(m - 1) + 1, 2(m - 1)) \\
(1, 2m) & \sim & ((m - 1)(m - 1) + 1, 2(m - 1)) \\
(0, 2(m - 1)) & \sim & (2(m - 1), 0) \\
(2(m - 1), 2(m - 1)) & \sim & (4(m - 1), 0) \\
\vdots & \vdots & \vdots \\
((m - 1)(m - 1), 2(m - 1)) & \sim & ((m + 1)(m - 1), 0) \\
((m + 1)(m - 1), 2(m - 1)) & \prec & (0, 2m)
\end{pmatrix}.$$  

### 3.5 An extension to $n$-dimensional spaces

The preceding results can be extended rather straightforwardly to $n$-dimensional Cartesian products. Thus, the $m$th-order cancellation axiom, the $(m + 1)$st-order symmetric cancellation axiom and unrestricted solvability w.r.t. the first component can be extended respectively as follows:
Axiom 9 \((C_m)\) for \(n\)-dimensional spaces) Let \(m \geq 2\). Let \((x_1^i, \ldots, x_n^i)\) and \((y_1, \ldots, y_n^i)\), \(i = 1, \ldots, m + 1\), be some elements of \(X\) such that \((y_1^i, \ldots, y_n^i)\) are permutations of \((x_1^j, \ldots, x_n^{j+1})\) for all \(j \in \{1, \ldots, n\}\). If \((x_1^1, \ldots, x_n^1) \succ (y_1^1, \ldots, y_n^1)\) for all \(i = 1, 2, \ldots, m\), then \((x_1^{m+1}, \ldots, x_n^{m+1}) \succ (y_1^{m+1}, \ldots, y_n^{m+1})\).

Axiom 10 \((S_{m+1})\) for \(n\)-dimensional spaces) Let \(m\) be an even number and let \(k = (m + 2)/2\). Let \(x_1^1, \ldots, x_1^{m+2} \in X_1\), and \(x_1^i, \ldots, x_i^{m+2}, y_1^i, \ldots, y_i^{m+2} \in X_i\), \(i = 2, \ldots, n\), be such that:

- for all \(j \leq k\) and all \(p > k\), \(x_1^j \neq x_1^p\);
- for all \(i \in \{2, \ldots, n\}\), there exists an index \(h_i \in \{0, \ldots, k\}\) such that
  \[x_1^i = y_1^i \iff \begin{cases} p = j & \text{if } j \leq h_i \text{ or if } k < j \leq k + h_i; \\ p = j + k & \text{if } h_i < j \leq k; \end{cases}\]
- at least one index \(h_i\) is equal to 0.

Then:
\[
\begin{pmatrix}
(x_1^1, x_2^1, \ldots, x_n^1) & \vdots & (x_1^k, y_1^1, \ldots, y_n^1) \\
(x_2^1, x_2^2, \ldots, x_n^2) & \vdots & (x_1^k, y_2^1, \ldots, y_n^2) \\
(x_1^3, x_2^3, \ldots, x_n^3) & \vdots & (x_1^k, y_3^1, \ldots, y_n^3) \\
& \vdots & \\
(x_1^k, x_2^k, \ldots, x_n^k) & \vdots & (x_1^k, y_1^k, \ldots, y_n^k) \\
(x_1^{k+1}, x_2^{k+1}, \ldots, x_n^{k+1}) & \vdots & (x_1^{k+1}, y_1^{k+1}, \ldots, y_n^{k+1}) \\
(x_1^{k+2}, x_2^{k+2}, \ldots, x_n^{k+2}) & \vdots & (x_1^{k+2}, y_1^{k+2}, \ldots, y_n^{k+2}) \\
& \vdots & \\
(x_1^{m+1}, x_2^{m+1}, \ldots, x_n^{m+1}) & \vdots & (x_1^{m+1}, y_1^{m+1}, \ldots, y_n^{m+1}) \\
& \vdots & \\
& \vdots & \\
\end{pmatrix} 
\succ (x_1^{m+2}, y_1^{m+2}, \ldots, x_n^{m+2}, y_n^{m+2}) \succ (x_1^{m+1}, y_1^{m+1}, \ldots, y_n^{m+1}).
\]

Axiom 11 (unrestricted solvability w.r.t. the first component) For all \(y \in X\) and all \(x_j \in X_j\), \(j = 2, \ldots, n\), there exists \(x_1 \in X_1\) such that \(y \sim (x_1, x_2, \ldots, x_n)\).

Theorem 1 is extended by the following theorem:

**Theorem 2** Let \(\succ\) be a weak order on \(X = \prod_{i=1}^n X_i\) and assume unrestricted solvability w.r.t. the first component (axiom 11). Then, for any \(m \geq 2\),
\[
\begin{cases}
\text{if } m \text{ is odd, then } (C_m) \iff (C_{m+1}) \\
\text{if } m \text{ is even, then } (C_m) \oplus (S_{m+1}) \iff (C_{m+1}).
\end{cases}
\]

The generic family of counterexamples of subsection 3.4 can also be extended to \(n\)-dimensional Cartesian products, thus showing that, in general, no cancellation axiom is sufficient to ensure additive representability:
Let $m$ be any odd number greater than or equal to 3. Let $\succeq$ be a weak order on $X = \mathbb{R} \times \{0, 2, 4, \ldots, 2m\} \times \{0, 2\}^{n-2}$ represented by the following utility function:

$$U(x_1, \ldots, x_n) = \begin{cases} H + x_1 + x_2 + \sum_{i=3}^{n} x_i & \text{if } x_2 < 2m, \\ V(x_1) + \sum_{i=3}^{n} x_i & \text{if } x_2 = 2m. \end{cases} \tag{5}$$

where

$$H = -m^2 - mn - 3n + 8.5,$$

$$V(x_1) = .5(x_1 \mod 2)^2 + 2|x_1/2|.$$

Then, the following lemma holds:

**Lemma 5** $(C_i)$ holds for all $i \in \{0, 1, \ldots, m + 1\}.$

However, $(C_{m+2})$ cannot hold because:

$$\begin{pmatrix}
(1 + 2(m + n - 3), 0, 0, \ldots, 0) & \sim & (1, 2(m - 1), 2, \ldots, 2) \\
(1 + 4(m + n - 3), 0, 0, \ldots, 0) & \sim & (1 + 2(m + n - 3), 2(m - 1), 2, \ldots, 2) \\
\vdots & \sim & \vdots \\
(1 + (m + 1)(m + n - 3), 0, 0, \ldots, 0) & \sim & (1 + (m - 1)(m + n - 3), 2(m - 1), 2, \ldots, 2) \\
(1, 2m, 0, \ldots, 0) & \sim & (1 + (m + 1)(m + n - 3), 2(m - 1), 2, \ldots, 2) \\
(0, 2(m - 1), 2, \ldots, 2) & \sim & (0, 2(m - 1), 2, \ldots, 2) \\
(2(m + n - 3), 2(m - 1), 2, \ldots, 2) & \sim & (4(m + n - 3), 0, 0, \ldots, 0) \\
\vdots & \sim & \vdots \\
((m - 1)(m + n - 3), 2(m - 1), 2, \ldots, 2) & \sim & ((m + 1)(m + n - 3), 0, 0, \ldots, 0) \\
((m + 1)(m + n - 3), 2(m - 1), 2, \ldots, 2) & < & (0, 2m, 0, \ldots, 0)
\end{pmatrix}.$$
Definition 1 (standard sequence w.r.t. the first component) For any set $M$ of consecutive integers, a set $\{x_k^1 \in X_1, k \in M\}$ is a standard sequence w.r.t. the first component if there exist $x_0^2, x_1^2 \in X_2$ such that $\text{Not}((x_1^1, x_0^2) \sim (x_1^1, x_1^2))$ and $(x_k^1, x_0^2) \sim (x_k^1, x_1^2)$ for all $k, k+1 \in M$. $\{x_0^1, x_1^2\}$ is called the mesh of the standard sequence.

Axiom 12 (Archimedean axiom w.r.t. the first component) Any bounded standard sequence w.r.t. the first component is finite, i.e., if $\{x_k^1 \in X_1, k \in M\}$ is a standard sequence with mesh $\{x_0^2, x_1^2\}$ and if there exist $y, z \in X$ such that $y \preceq (x_k^1, x_2^0) \preceq z$ for all $k \in M$, then $M$ is finite.

Obviously independence and the Archimedean axiom are required. However they are still not sufficient. Indeed, assume that $X = \mathbb{C} \times \{0,1\}$, where $\mathbb{C}$ is the set of complex numbers. Let $\succ$ on $X$ be such that

- for all $x_1, y_1 \in \mathbb{C}$, $(x_1,0) \prec (y_1,1)$,
- for all $x_1 + iy_1, z_1 + it_1 \in \mathbb{C}$ and all $x_2 \in \{0,1\}$, $(x_1 + iy_1, x_2) \succ (z_1 + it_1, x_2) \iff x_1 \geq z_1$ or $(x_1 = z_1$ and $y_1 \geq t_1$).

Then $\succ$ is not even representable by a utility function, it is not a fortiori representable by an additive utility function. Hence, we will also require that $\succ$ be representable by a utility function on $X$. This set of axioms is not yet sufficient to ensure additive representability because the structure of the second component is too poor. Hence we also require that $X_2$ be equally spaced and the lemma below follows.

Definition 2 $X_2$ is equally spaced if, for all $x_1, y_1 \in X_1$ and all $i, i+1, i+2 \in N$:

$$(x_1, x_2^i) \sim (y_1, x_2^{i+1}) \iff (x_1, x_2^{i+1}) \sim (y_1, x_2^{i+2}).$$

Lemma 6 Let $\succ$ be a weak order on $X = X_1 \times X_2$, where $X_2 = \{x_i^2, i \in N\}$. Assume restricted solvability w.r.t. $X_1$ (axiom 6), independence (axiom 1) and the Archimedean axiom (axiom 12). Suppose that $\succ$ is representable by a utility function, that $x_2^{i+1} \succeq x_2^i$ for all $i, i+1 \in N$, and that $X_2$ is equally spaced (definition 2). Then $\succ$ is representable by an additive utility function.\footnote{This lemma is important because it can serve as a basis to prove the classical representation theorem for two-dimensional Cartesian products — see Fishburn (1970, theorem 5.2, p58) — using neither topology nor algebra. But this is beyond the scope of this paper and is explained in full detail in Gonzales (1996b).}

Lemma 6 can be viewed as a slight generalization of Wakker (1991): in addition to equal spacedness\footnote{In fact, the definition of equal spacedness stated in Wakker (1991) is different from definition 2, but together with restricted solvability and weak separability (which are assumed by Wakker (1991)) both definitions are equivalent.} w.r.t. every component of $X$, the latter indeed requires restricted solvability w.r.t. every component.

Of course, the existence of a utility function representing $\succ$ is not a very appealing assumption. However, according to a well known theorem (see Debreu (1954, 1964) or Krantz et al. (1971, p.40)), if there exists a set $A \subset X$, at most denumerable, such that $x \prec y \Rightarrow$ there exist $a, b \in A$ such that $x \preceq a \prec b \preceq y$, then $\succ$ is representable by a
utility function. In the present framework, the assumption that there exists a utility function representing \( \succeq \) can be substituted by the following condition:

there exists a denumerable set \( A_1 \subset X_1 \) such that
\[
x_1 \prec y_1 \implies \text{there exist } a_1, b_1 \in A_1 \text{ such that } x_1 \prec a_1 \prec b_1 \prec y_1.
\]

As was mentioned in Wakker (1991), although equal spacedness “may have been considered complicated because it is of a combinatorial nature”, representation theorems are in fact simpler than in the classical frameworks (in Krantz et al. (1971) and Fishburn (1970), the Thomsen condition must be assumed to derive the existence of additive utilities in two-dimensional Cartesian products).

5 Appendix: Proofs

Proof of lemma 1: It is straightforward to see that \((C_2)\) implies independence and the generalized Thomsen condition. Let us prove the converse. \((C_2)\) can be written as:

\[
\begin{align*}
(x_1, x_2) &\succ (x_1', x_2') \\
(y_1, y_2) &\succ (y_1', y_2')
\end{align*}
\]

\[
\implies (z_1, z_2) \succ (z_1', z_2'),
\]

where \((x_i', y_i', z_i')\) are permutations of \((x_i, y_i, z_i)\) for \(i \in \{1, 2\}\). By permutation of the above left hand side (LHS) relations, it is not restrictive to assume that either \(x_2'\) or \(y_2'\) is equal to \(x_2\). Assume that \(y_2' = x_2\). If \(y_1 = x_1'\), then the right hand side (RHS) relation is deduced from the generalized Thomsen condition. If \(y_1 = y_1'\), then substituting \(y_1\) and \(y_1'\) by \(y_1\) and applying the generalized Thomsen condition and then independence gives the RHS relation. If \(y_1 = z_1'\), then either \(x_1 = x_1'\) and, substituting \(x_1\) and \(x_1'\) by \(y_1\) and applying the generalized Thomsen condition and independence, one gets the RHS relation, or \(x_1 = y_1'\) and, by transitivity, one gets the RHS relation. A similar reasoning holds when \(x_2' = x_2\). Hence, the conjunction of independence and the generalized Thomsen condition implies the second order cancellation axiom.

Proof of lemma 2: By lemma 1, to prove lemma 2 it is sufficient to show that independence (axiom 1), unrestricted solvability w.r.t. the first component (axiom 5) and the Thomsen condition (axiom 2) imply the generalized Thomsen condition (axiom 3). Suppose that the generalized Thomsen condition does not hold. Then, there exist \(x_1, y_1, z_1 \in X_1\) and \(x_2, y_2, z_2 \in X_2\) such that \((x_1, z_2) \succ (z_1, y_2), (z_1, x_2) \succ (y_1, z_2)\) and \((y_1, y_2) \succ (x_1, x_2)\), with at least one strict preference relation. By unrestricted solvability w.r.t. the first component, there exists \(t_1 \in X_1\) such that \((x_1, z_2) \sim (t_1, y_2)\), and by transitivity, \((t_1, y_2) \succ (z_1, y_2)\). Therefore, by independence, \((t_1, x_2) \succ (z_1, x_2) \succ (y_1, z_2)\). Hence \((x_1, z_2) \sim (t_1, y_2), (t_1, x_2) \succ (y_1, z_2)\) and \((y_1, y_2) \succ (x_1, x_2)\), with at least one strict preference relation. Again, by unrestricted solvability w.r.t. the first component, there exists \(s_1 \in X_1\) such that \((t_1, x_2) \sim (s_1, z_2)\) and

\[
(x_1, z_2) \sim (t_1, y_2), \quad (t_1, x_2) \sim (s_1, z_2) \quad \text{and} \quad (s_1, y_2) \succ (x_1, x_2).
\]

But the above implication is impossible because it contradicts the Thomsen condition (axiom 2). Hence, by contradiction, it entails lemma 2. ■
Proof of lemma 3: Assume that (2) holds. Let \( x_1, x_2 \in X_1 \) and \( x_2, x_3 \in X_2 \) be arbitrary elements such that \( (x_1, x_2) \sim (x_1, x_2) \) and \( (x_1, x_3) \sim (x_3, x_2) \). By symmetry, it is not restrictive to assume that \( x_2 \preceq x_3 \).

Let us show that \( (x_1, x_2) \sim (x_1, x_2) \). If \( x_2 \preceq x_2 \preceq x_3 \), then, by (2), \( (x_1, x_2) \sim (x_1, x_2) \). Now assume that either \( x_2 \preceq x_3 \preceq x_2 \) or \( x_2 \preceq x_2 \preceq x_3 \), and that
\[
\text{Not}(x_2, x_2) \sim (x_1, x_2)). \tag{8}
\]

By unrestricted solvability w.r.t. the first component, there exists \( g^1 \in X_1 \) such that \( (g^1, x_2) \sim (x_1, x_2) \) and, by (8),
\[
\text{Not}(g^1, x_2) \sim (x_1, x_2)). \tag{9}
\]

But, then, \( (x_2, x_2) \sim (g^1, x_2), (x_2, x_2) \sim (x_1, x_2) \) and either \( x_3 \preceq x_2 \preceq x_2 \) or \( x_2 \preceq x_2 \preceq x_3 \). Hence, by (2), \( (g^1, x_2) \sim (x_1, x_3), \) and so \( (g^1, x_2) \sim (x_1, x_2) \). But this is impossible according to (9) and independence. Hence (8) cannot hold. \( \blacksquare \)

Proof of theorems 1 and 2: Theorem 1 is proved when fixing \( n = 2 \). \((C_m+1) \Rightarrow (C_m)\) is a classical result and since \( (S_m+1) \) is a subset of \((C_m+1)\), the implication from right to left is obvious. The proof of the converse is adapted from a proof given in Jaffray (1974a) and Jaffray (1992). We first note that \((C_m)\) can be stated as follows:

**Condition \((C_m)\):** Suppose that, for all \( j \in \{1, \ldots, n\} \), \( (x_j, \ldots, x_{j+m+1}) \) is a permutation of \((y_j, \ldots, y_{j+m+1}) \). Then \( [(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n) \text{ for all } i \in \{1, \ldots, m+1\}] \Rightarrow [(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n) \text{ for all } i \in \{1, \ldots, m+1\}] \).

Hereafter we will use this definition instead of axiom 9 in subsection 3.5. Suppose that \((C_m)\) holds—as well as \((S_m+1)\) if \( m \) is even—and consider the following sequence of preferences:
\[
[(x_1, \ldots, x_n) \succ (y_1, \ldots, y_n) \text{ for all } i \in \{1, \ldots, m+2\}] \tag{10}
\]

**First case:** there exists \( i_0 \) and \( j_0 \) such that \( x^{i_0} = y^{j_0} \):

Substitute \( x^{i_0} \preceq y^{i_0} \) and \( x^{j_0} \preceq y^{j_0} \) by \( x^{i_0} \preceq y^{j_0} \) and \( x^{i_0} \preceq y^{j_0} \). Since \( x^{i_0} = y^{j_0} \), by removing the line \( x^{i_0} \preceq y^{j_0} \), the conditions of application of \((C_m)\) obtain, and \( x^{j_0} \preceq y^{i_0} \) and \( x^i \sim y^i \) for all \( i \neq i_0, j_0 \). But \( x^{j_0} \preceq y^{j_0} \preceq x^{i_0} \preceq y^{j_0} \), hence \( x^{i_0} \sim y^{j_0} \) and \( x^{j_0} \sim y^{j_0} \). Therefore \((C_m+1)\) holds.

If the previous case does not obtain, change the order of the preference relations so as to get cycles w.r.t. the first component, i.e., a sequence of the form:
\[
(x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n), \quad (x_1, x_2, \ldots, x_n) \succ (y_1, y_2, \ldots, y_n) \quad \text{for all } i \in \{2, \ldots, k\} \tag{11}
\]

**Second case:** \( k = m + 2 \):

We know that there exists \( i_2 \) such that \( y_i^{i_2} = x_i^{i_2} \). If \( i_2 \neq 1 \) then, by unrestricted solvability w.r.t. \( X_1 \), there exists \( g^{i_2-1}(2) \) such that \( (g^{i_2-1}(2), y_i^{i_2-1}, y_i^{i_2}, \ldots, y_n^{i_2}) \sim (x_1^{i_2-1}, y_i^{i_2}, \ldots, y_n^{i_2}) \). But then, by \((C_2)\), the following holds:
reaches the first preference relation. Thus we get a cycle of the following form:

\[
(g_{i_2-1}^{i_2-1}(2), y_2^{i_2-1}, y_3^{i_2-1}, \ldots, y_n^{i_2-1}) \sim (x_1^{i_2-1}, y_2^{i_2}, \ldots, y_n^{i_2-1})
\]

\[
(x_1^{i_2-2}, x_2^{i_2-1}, x_3^{i_2-1}, \ldots, x_n^{i_2-1}) \succ (x_1^{i_2-2}, y_2^{i_2-1}, \ldots, y_n^{i_2-1})
\]

\[
\downarrow
\]

\[
(x_1^{i_2-2}, y_2^{i_2}, y_3^{i_2-1}, \ldots, y_n^{i_2-1}) \succ (g_{i_2-1}^{i_2-1}(2), x_2^{i_2-1}, \ldots, x_n^{i_2-1}).
\]

So, the lines corresponding to indices \( i_2 - 1 \) and \( i_2 \) in (11) can be substituted by

\[
(g_{i_2-1}^{i_2-1}(2), x_2^{i_2-1}, x_3^{i_2-1}, \ldots, x_n^{i_2-1}) \succ (x_1^{i_2-2}, y_2^{i_2}, y_3^{i_2-1}, \ldots, y_n^{i_2-1})
\]

\[
(x_1^{i_2}, x_2^{i_2}, x_3^{i_2}, \ldots, x_n^{i_2}) \succ (g_{i_2-1}^{i_2-1}(2), y_2^{i_2-1}, y_3^{i_2}, \ldots, y_n^{i_2}).
\]

Repeat the process with indices \( i_2 - 1, i_2 - 2, \ldots, 1 \), so that \( y_2^{i_2} \) moves upward till it reaches the first preference relation. Thus we get a cycle of the following form:

\[
(g^{1}(2), x_2^{1}, \ldots, x_n^{1}) \succ (x_1^{m+2}, x_2^{1}, x_3^{1}, \ldots, y_n^{1}),
\]

\[
(g^{i}(2), x_2^{i}, \ldots, x_n^{i}) \succ (g^{i-1}(2), y_2^{\sigma_2(1)}, y_3^{1}, \ldots, y_n^{i}) \quad \text{for all } i \in \{2, \ldots, m + 2\},
\]

where \( \sigma_2 \) is the permutation such that \( \sigma_2(i) = i - 1 \) for all \( i \leq i_2 \) and \( \sigma_2(i) = i \) for all \( i > i_2 \). Similarly, there exists \( i_3 \) such that \( y_3^{i_3} = x_3^{1} \). Repeat with the above cycle the same process, i.e., find elements \( g^{(3)} \) to move \( y_3^{i_3} \) upward to the first preference relation. By induction, repeat the process for all \( j \geq 4 \), hence resulting in the cycle:

\[
(g^{1}(n), x_2^{1}, \ldots, x_n^{1}) \succ (x_1^{m+2}, x_2^{1}, x_3^{1}, \ldots, x_n^{1}),
\]

\[
(g^{j}(n), x_2^{i}, \ldots, x_n^{i}) \succ (g^{j-1}(n), y_2^{\sigma_j(1)}, y_3^{\sigma_j(1)}, \ldots, y_n^{\sigma_j(1)}) \quad \text{for all } i \in \{2, \ldots, m + 2\},
\]

where \( \sigma_j(i) = i - 1 \) for all \( i \leq i_j \) and \( \sigma_j(i) = i \) for all \( i > i_j \). The formulas for modifying the \((r-1)\)th and the \( r \)th lines of the \( p \)th cycle constructed are as follows: (i) before the modification, the cycle is
By induction this leads to replacing all the $r$'s in (11) by $\geq$'s. Hence (ii) is generalized by:

\[
(g^{r-1}(p), y_2^{r-1}, \ldots, y_p^{r-1}, y_{p+1}, \ldots, y_n) \geq (g^{r-2}(p-1), y_2^{r-1}, \ldots, y_p^{r-1}, y_{p+1}, \ldots, y_n)
\]

and (iv) the $(r-1)$th and rth lines of the cycle are substituted by:

\[
(g^{r-2}(p-1), y_2^{r-1}, \ldots, y_p^{r-1}, y_{p+1}, \ldots, y_n) \geq (g^{r-1}(p), x_2^{r-1}, \ldots, x_n^{r-1}).
\]

Now, by independence, the first line of (14) can be substituted by $(g^1(n), x_2^{m+2}, \ldots, x_n^{m+2}) \geq (x_1^{m+2}, x_2^{m+2}, \ldots, x_n^{m+2})$. Then the first case of this proof can be applied and $\geq$'s can be replaced by $\sim$'s in (14). Now to propagate indifference relations back to the initial cycle (11), consider the converse of (12), namely, for the $(r-1)$th and rth lines of the cycle generated by the $g^{(i)}(p)$, the following implications:

\[
(g^{r-1}(p), x_2^{r-1}, \ldots, x_n^{r-1}) \sim (g^{r-2}(p-1), y_2^{r-1}, \ldots, y_p^{r-1}, y_{p+1}, \ldots, y_n^{r-1})
\]

and

\[
(g^{r-1}(p), y_2^{r-1}, \ldots, y_p^{r-1}, y_{p+1}, \ldots, y_n^{r-1}) \sim (g^{r-2}(p-1), x_2^{r-1}, \ldots, x_n^{r-1}).
\]

But, then the $(r-1)$th and rth lines can be substituted back to:

\[
(g^{r-1}(p-1), x_2^{r-1}, \ldots, x_n^{r-1}) \sim (g^{r-2}(p-1), y_2^{r-1}, \ldots, y_p^{r-1}, y_{p+1}, \ldots, y_n^{r-1}),
\]

\[
(g^{r}(p), x_2^{r}, \ldots, x_n^{r}) \sim (g^{r-1}(p-1), y_2^{r}, \ldots, y_p^{r}, y_{p+1}, \ldots, y_n^{r}).
\]

By induction this leads to replacing all the $\geq$'s in (11) by $\sim$'s. Hence $(C_{m+1})$ holds.

**Third case:** there exist exactly 2 cycles of length $k = (m + 2)/2$:
If there exist \( i, j \) such that \( i \leq \frac{m+2}{2} < j \) and \( x^i_1 = x^j_1 \), then the second case can be applied since both cycles can be aggregated to form one cycle of length \( m + 2 \). Of course, \( x^j_1 \) appears in several lines, but the process described in the second case can still be applied, hence ensuring that \((C_{m+1})\) holds.

Assume now that there exist no \( i, j \) such that \( i \leq \frac{m+2}{2} < j \) and \( x^i_1 = x^j_1 \). Using transformation (12), each element \( y^j_{i} \), \( i \leq k \), such that there exists \( p \leq k \) such that \( x^p_j = y^j_{i} \), can be moved in the list (11) such that \( x^p_j = y^j_{i} \). A transformation similar to (12) would enable to move \( x^p_j \) along the list instead of \( y^j_{i} \).

If for all \( j \in \{2, \ldots, n\} \), there exist \( i, p \leq k \) such that \( x^p_j = y^j_{i} \), then transformations described above enable to modify the initial list (11) so that \( x^j_1 = y^j_{i} \) for all \( j \). Thus, the first case of this proof can be applied and \( \gtrsim \)'s can be substituted by \( \sim \)'s. Inverse transformations entail that this result also holds for the initial list, so that \((C_{m+1})\) holds.

If there exists an index \( j \in \{2, \ldots, n\} \) such that, for all \( i \leq k \), there exists no \( p \leq k \) such that \( x^p_j = y^j_{i} \), then we are in the conditions of the \((m+1)\)st order symmetric cancellation axiom, and so \((C_{m+1})\) holds.

**Fourth case: there exists a cycle of length \( k < \frac{m+2}{2} \):**

Suppose that \( y_{2}^{k+1} = x^3_2 \) and that the corresponding relation is \((x^{k+1}_1, x^{k+1}_2, x^{k+1}_3, \ldots, x^{k+1}_n) \gtrsim (x^3_1, x^3_2, x^3_3, \ldots, y^{k+1}_n)\). Then, by unrestricted solvability w.r.t. the first component, there exists \( g^1 \in X_1 \) such that \((g^1, x^{k+1}_1, x^{k+1}_2, x^{k+1}_3, \ldots, x^{k+1}_n) \sim (x^3_1, x^3_2, x^3_3, \ldots, y^{k+1}_n)\).

Again, by unrestricted solvability w.r.t. the first component, there exist \( g^2, \ldots, g^k \) such that, for all \( i \in \{1, \ldots, k-1\} \), \((g^i, y^i_2, y^i_3, \ldots, y^i_n) \sim (g^{i+1}, x^{i+1}_2, x^{i+1}_3, \ldots, x^{i+1}_n)\) and \((C_{2k-1})\) implies that \((g^k, y^k_2, y^k_3, \ldots, y^k_n) \gtrsim (g^1, x^{k+1}_1, x^{k+1}_2, x^{k+1}_3, \ldots, x^{k+1}_n) \gtrsim (x^{k+1}_1, x^{k+1}_2, x^{k+1}_3, \ldots, x^{k+1}_n)\).

So one can substitute in the initial list the lines \((g^1, x^1_2, x^1_3, \ldots, x^1_n) \gtrsim (g^k, y^k_2, y^k_3, \ldots, y^k_n)\) and \((x^1_1, x^1_2, x^1_3, \ldots, x^1_n) \gtrsim (x^k_1, x^k_2, x^k_3, \ldots, y^k_n)\) by \((x^{k+1}_1, x^{k+1}_2, x^{k+1}_3, \ldots, x^{k+1}_n) \gtrsim (g^k, y^k_2, y^k_3, \ldots, y^k_n)\) and \((g^1, x^1_2, x^1_3, \ldots, x^1_n) \sim (x^k_1, x^k_2, x^k_3, \ldots, y^k_n)\).

Hence cycle \((1 \to k)\) — where \((1 \to k)\) denotes the list of relations located on rows 1 through \( k \) — and cycle \((k + 1 \to p)\) now form only one cycle \( 1 \to p \). One can go on with the same process until the initial list (11) is transformed into a list with a cycle of length \( m + 2 \) (with some \( x^1_1 \) possibly in several lines), or is transformed into a list containing two cycles of length \( \frac{m+2}{2} \) (with some \( x^1_1 \) possibly in several relations). The former corresponds to case 2 and so all the \( \gtrsim \)'s can be substituted by \( \sim \)'s and one can trace the \( \sim \)'s back to the initial list. The latter case can be split into two cases: either no cycle contains several lines with the same \( x^1_1 \), and we are in the conditions of the third case, or at least one cycle contains several identical \( x^1_1 \)'s.

So now we have two cycles of length \( \frac{m+2}{2} \), i.e., cycles \((1 \to k)\) and \((k + 1 \to m + 2)\), of which at least one contains several identical \( x^1_1 \)'s. Consider that the latter is cycle \((1 \to k)\). If the identical \( x^1_1 \)'s are not on a same line, i.e., the cycle \((1 \to k)\) can be split into several cycles of length strictly shorter than \( \frac{m+2}{2} \), then the process described above can be used to aggregate one of these small cycles with the second cycle of length \( \frac{m+2}{2} \), hence creating a new cycle of overall length \( m + 2 \); then using case 2, we can deduce that \((C_{m+1})\) holds. If all the identical \( x^1_1 \)'s are on the same lines, i.e., on some lines like

\[
(g^i, x^p_i, x^p_{i+1}, \ldots, x^p_n) \gtrsim (g^j, y^h_j, y^h_{j+1}, \ldots, y^h_n),
\]

(15)
then, removing these lines from the cycle $1 \rightarrow k$, we get a cycle of length $< \frac{m+2}{2}$, that we can aggregate with cycle $k+1 \rightarrow m+2$. Then, using independence, lines (15) can be substituted by $(x_1^{m+2}, x_2^p, x_3^p, \ldots, x_n^p) \geq (x_1^{m+2}, y_2^p, y_3^p, \ldots, y_n^p)$. But then we have created a cycle of length $m+2$ and the second case ensures that $(C_{m+1})$ holds. ■

**Proof of lemmas 4 and 5:** Lemma 4 is proved by assigning $n = 2$ in the rest of the proof. The proof is organized in three steps: first, it is shown that independence holds; then the second order cancellation axiom is shown to hold, and, finally, using theorem 2, the other cancellation axioms are studied.

**First step: independence holds**

Clearly, by definition of $U$, independence holds w.r.t. components 3 to $n$. Independence also holds w.r.t. the second component: suppose that $(x_1, 2m, x_3, \ldots, x_n) \geq (y_1, 2m, y_3, \ldots, y_n)$. Then, by unrestricted solvability w.r.t. the first component, there exists $g_1$ such that $(g_1, 2m, x_3, \ldots, x_n) \sim (y_1, 2m, y_3, \ldots, y_n)$ or, equivalently:

**Proof:**

\[
0.5 \left( g_1 \mod 2 \right)^2 + 2 \left\lfloor \frac{g_1}{2} \right\rfloor + \sum_{i=3}^{n} x_3 = 0.5 \left( y_1 \mod 2 \right)^2 + 2 \left\lfloor \frac{y_1}{2} \right\rfloor + \sum_{i=3}^{n} y_3
\]

and so $0.5 (g_1 \mod 2)^2 - 0.5 (y_1 \mod 2)^2 = 2 \left\lfloor \frac{g_1}{2} \right\rfloor + 2 \left\lfloor \frac{y_1}{2} \right\rfloor + \sum_{i=3}^{n} (y_3 - x_3)$. The right hand side being a multiple of 2, it is clear that there exists an integer $k$ (positive or negative) such that $y_1 = g_1 + 2k$. But then

\[
-2 \left\lfloor \frac{g_1}{2} \right\rfloor + 2 \left\lfloor \frac{y_1}{2} \right\rfloor + \sum_{i=3}^{n} (y_3 - x_3) = 0 = -g_1 + y_1 + \sum_{i=3}^{n} (y_3 - x_3),
\]

or, equivalently, for all $x_2 \in \{0, \ldots, 2(m-1)\}$, $g_1 + x_2 + \sum_{i=3}^{n} x_3 + H = y_1 + x_2 + \sum_{i=3}^{n} y_3 + H$. Therefore, since $U$ represents $\geq$, $(g_1, x_2, x_3, \ldots, x_n) \sim (y_1, x_2, y_3, \ldots, y_n)$, and since $0.5 (x_1 \mod 2)^2 + 2 \left\lfloor x_1/2 \right\rfloor \geq 0.5 (g_1 \mod 2)^2 + 2 \left\lfloor g_1/2 \right\rfloor$, $(x_1, x_2, x_3, \ldots, x_n) \geq (y_1, x_2, y_3, \ldots, y_n)$.

Conversely, if $(g_1, x_2, x_3, \ldots, x_n) \sim (y_1, x_2, y_3, \ldots, y_n)$, $x_2 \neq 2m$, then, since all the $x_i$’s and all the $y_i$’s, $i \geq 2$, are multiples of 2, there exists an integer $k$ (positive or negative) such that $g_1 = y_1 + 2k$. Then $0.5 (g_1 \mod 2)^2 + 2 \left\lfloor g_1/2 \right\rfloor + \sum_{i=3}^{n} x_3 = 0.5 (y_1 \mod 2)^2 + 2 \left\lfloor y_1/2 \right\rfloor + \sum_{i=3}^{n} y_3$ and so $(g_1, 2m, x_3, \ldots, x_n) \sim (y_1, 2m, y_3, \ldots, y_n)$.

Independence axiom also holds w.r.t. $X_1$. Indeed, assume that $(x_1, x_2, x_3, \ldots, x_n) \geq (y_1, y_2, y_3, \ldots, y_n)$. If $x_2 = y_2$ or if $x_2 \neq 2m$ and $y_2 \neq 2m$, then, clearly, by definition of $U$, $(y_1, x_2, x_3, \ldots, x_n) \sim (y_1, y_2, y_3, \ldots, y_n)$ for any $y_1 \in \mathbb{R}$. Now, $y_2 = 2m$ and $x_2 \neq 2m$ is impossible because $0.5 (x_1 \mod 2)^2 + 2 \left\lfloor x_1/2 \right\rfloor + \sum_{i=3}^{n} y_3 \geq x_1 - 0.5$ and

\[
\sum_{i=1}^{n} x_i + H \leq x_1 + 2(m-1) + 2(n-2) + H \leq x_1 - (m-1)^2 - n(m+1) + 3.5
\]

and so, since $m \geq 3$ and $n \geq 2$, $\sum_{i=1}^{n} x_i + H \leq x_1 - 8.5$. Therefore, assume that $(x_1, 2m, x_3, \ldots, x_n) \geq (x_1, y_2, y_3, \ldots, y_n)$. Then, clearly, for any $y_1 \in \mathbb{R}$, $0.5 (y_1 \mod 2)^2 + 2 \left\lfloor y_1/2 \right\rfloor + \sum_{i=3}^{n} y_3 \geq \sum_{i=1}^{n} y_i + H$, and so $(y_1, 2m, x_3, \ldots, x_n) \sim (y_1, y_2, y_3, \ldots, y_n)$. Hence independence holds.
Second step: \((C_2)\) holds

Assume that \((C_2)\) does not hold, i.e., that there exist \(x_i^j, y_i^j, i \in \{1, \ldots, n\}, j \in \{1, 2, 3\}\), such that \((y_i^j, y_i^j, y_i^j)\) are permutations of \((x_i^1, x_i^2, x_i^3)\) and such that

\[
\frac{(x_1^1, x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^1, y_1^2, y_1^3, \ldots, y_1^n)} \succ \frac{(x_1^2, x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^2, y_1^2, y_1^3, \ldots, y_1^n)} \succ \frac{(x_1^3, x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^3, y_1^2, y_1^3, \ldots, y_1^n)}
\]

(16)

with at least one strict preference \(\succ\). Let us prove that (16) cannot hold. If \(x_i^j = 2m\) for all \(j \in \{1, 2, 3\}\), then, by independence, one can substitute all the \(x_i^j, y_i^j\) by 0; translating these preference relations in terms of utility \(U\), and summing the three inequalities thus obtained, one gets \(\sum_{j=1}^{n} (\sum_{i=1}^{n} x_i^j + H) > \sum_{j=1}^{n} (\sum_{i=1}^{n} y_i^j + H)\). But the \(y_i^j\)'s are permutations of the \(x_i^j\)'s, so that the last inequality cannot hold. Similarly, if \(x_i^j \neq 2m\) for all \(j \in \{1, 2, 3\}\), then (16) cannot hold.

If there exist \(i, j, i \neq j\), such that \(x_i^j = x_i^j = 2m\), then at least one preference relation has a \(2m\) term on its right and left hand sides. By independence, one can substitute the \(2m\)'s by 0's. So, one need only study the following case: there exists a unique index \(i\) such that \(x_i^j = 2m\), and, moreover, \(y_i^j \neq 2m\). Without loss of generality, by reordering the preference relations, \(x_1^2 = 2m\) and \(y_2^2 = 2m\). Hence the following relations:

\[
\frac{(x_1^1, x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^1, y_1^2, y_1^3, \ldots, y_1^n)} \succ \frac{(x_1^2, x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^2, 2m, y_1^3, \ldots, y_1^n)} \succ \frac{(x_1^3, x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^3, y_2^2, y_1^3, \ldots, y_1^n)}
\]

with at least one strict preference \(\succ\).

If \(y_1^2 = x_1^1\), then, summing the utilities over the three preference relations, one gets:

\[
0.5 (x_1^1 \mod 2)^2 + 2|x_1^1/2| + x_1^2 + x_1^3 - x_1^2 + \sum_{j=1}^{3} (\sum_{i=2}^{n} x_i^j) + 2H >
0.5 (x_1^1 \mod 2)^2 + 2|x_1^1/2| + x_1^2 + x_1^3 - x_1^2 + \sum_{j=1}^{3} (\sum_{i=2}^{n} x_i^j + 2H).
\]

Hence \(y_1^2 = x_1^1\) is impossible. If \(y_1^2 = x_1^2\), then, translating the second preference relation into its utility counterpart, one gets:

\[
x_1^2 + x_1^3 + \sum_{i=3}^{n} x_i^3 + H \geq V(x_1^2) + \sum_{i=3}^{n} y_i^2.\]

But this is impossible because \(V(x_1^2) \geq x_1^2 - 0.5\) and, since \(m \geq 3\) and \(n \geq 2\), \(x_1^2 + x_1^3 + \sum_{i=3}^{n} x_i^3 + H \leq x_1^2 - 8.5\). Hence \(y_1^2 = x_1^3\). Consequently, if (16) holds, then it can be written as:

\[
\frac{(x_1^1, 2m, x_1^3, \ldots, x_1^n)}{(y_1^1, y_1^2, y_1^3, \ldots, y_1^n)} \succ \frac{(x_1^2, x_1^3, \ldots, x_1^n)}{(y_1^3, y_1^2, y_1^3, \ldots, y_1^n)}
\]

with at least one strict preference \(\succ\).

If \(y_1^3 = x_1^3\), then, in terms of utility \(U\), the last two preference relations are equivalent to

\[
x_1^3 \geq V(x_1^3) + \sum_{i=3}^{n} (y_i^3 - x_i^3) - x_2^2 - H \text{ and } x_1^3 \geq x_1^3 + \sum_{i=2}^{n} (y_i^3 - x_i^3).
\]

Therefore

\[
x_1^3 \geq V(x_1^3) + \sum_{i=3}^{n} (y_i^3 - x_i^3) + \sum_{i=3}^{n} (y_i^3 - x_i^3) + y_2^3 - x_2^3 - x_2^3 - H.
\]

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But since the \( y_i \)'s are permutations of the \( x_i \)'s, \( \sum_{i=3}^{n}(y_i^2 - x_i^2) + \sum_{i=3}^{n}(y_i^3 - x_i^3) = \sum_{i=3}^{n}(x_i^2 - y_i^2) \). And since \( m \geq 3 \) and \( n \geq 2 \),

\[
-H = \left( (m - 2)^2 + n(m + 1) - 4.5 \right) + 2(n - 2) + 4(m - 1)
\]

\[
\geq \sum_{i=3}^{n}(y_i^1 - x_i^1) + x_2^2 + x_2^2 + \left[ (m - 2)^2 + n(m + 1) - 4.5 \right]
\]

\[
\geq \sum_{i=3}^{n}(y_i^1 - x_i^1) + x_2^2 + x_2^2 + 4.5,
\]

one gets: \( x_1^3 \geq V(x_1^3) + 4.5 \), which is impossible since \( V(x_1^3) \geq x_1^3 - 0.5 \). Therefore, \( y_i^1 = x_1^1 \) and (16) can be written as:

\[
\left( (x_1^1, 2m, x_3^3, \ldots, x_n^3) \right) \sim (x_1^2, y_2^1, y_3^1, \ldots, y_n^1)
\]

\[
\left( x_1^2, x_2^2, x_3^3, \ldots, x_n^3 \right) \sim (x_1^2, 2m, y_3^2, \ldots, y_n^2)
\]

\[
\left( x_1^3, x_2^3, x_3^3, \ldots, x_n^3 \right) \sim (x_1^3, y_2^3, y_3^3, \ldots, y_n^3)
\]

with at least one strict preference \( \succ \).

Now, by unrestricted solvability w.r.t. the first component, there exists \( g^2 \in \mathbb{R} \) such that \( (g_2^2, x_2^3, \ldots, x_2^3) \sim (x_1^1, 2m, y_3^3, \ldots, y_n^3) \) and there exists \( g_1^2 \in \mathbb{R} \) such that \( (g_1^2, 2m, x_3^3, \ldots, x_n^3) \sim (g_2^2, y_3^3, y_3^3, \ldots, y_n^3) \). But then,

\[
\left( (g_1^2, 2m, x_3^3, \ldots, x_n^3) \right) \sim (g_2^2, y_3^3, y_3^3, \ldots, y_n^3)
\]

\[
\left( g_2^2, x_2^3, x_3^3, \ldots, x_n^3 \right) \sim (x_1^3, 2m, y_3^3, \ldots, y_n^3)
\]

\[
\left( x_1^3, x_2^3, x_3^3, \ldots, x_n^3 \right) \sim (g_1^2, y_3^3, y_3^3, \ldots, y_n^3)
\]

Therefore, in terms of utility function \( U \),

\[
V(g_1^1) + \sum_{i=3}^{n} x_i^1 = g^2 + y_1^1 + \sum_{i=3}^{n} y_i^1 + H \quad \text{and} \quad g^2 + x_2^2 + \sum_{i=3}^{n} x_i^2 + H = V(x_1^3) + \sum_{i=3}^{n} y_i^2.
\]

Replacing in the first equality \( g^2 \) by its value given by the second equality, \( V(g_1^1) + \sum_{i=3}^{n} x_i^1 = V(x_1^3) + \sum_{i=3}^{n} (y_i^2 - x_i^1) - x_2^2 + y_2^1 + \sum_{i=3}^{n} y_i^1 \) or, equivalently,

\[
V(g_1^1) = V(x_1^3) + \sum_{i=3}^{n} (y_i^2 - x_i^1) - x_2^2 + y_2^1 + \sum_{i=3}^{n} (y_i^1 - x_i^1).
\]

Now, since the \( y_i^j \)’s are permutations of \( x_i^j \)'s, \( 2m + x_2^2 + x_3^2 = y_3^2 + 2m + y_2^2 \) and \( \sum_{i=3}^{n} (y_i^1 - x_i^1) + \sum_{i=3}^{n} (y_i^2 - x_i^2) + \sum_{i=3}^{n} (y_i^3 - x_i^3) = 0 \). So, (17) is equivalent to

\[
V(g_1^1) + y_3^2 + \sum_{i=3}^{n} y_i^3 = V(x_1^3) + x_2^2 + \sum_{i=3}^{n} x_i^3.
\]

Since \( x_i^j \) and \( y_i^j, i \geq 2 \), are multiples of 2, there exists an integer \( k \) (positive or negative) such that \( g_1^1 = x_1^3 + 2k \) and \( g_1^1 + y_3^2 + \sum_{i=3}^{n} y_i^3 = x_1^3 + x_2^2 + x_3^3 + \sum_{i=3}^{n} x_i^3 \). But this equality corresponds to the preference relation \( (g_1^1, y_2^2, \ldots, y_n^3) \sim (x_3^1, x_2^3, \ldots, x_n^3) \). So (16) cannot occur and, consequently, \((C_2)\) holds.
Third step: \((C_i), \ i \geq 3, \) holds

According to theorem 2, it is sufficient to show that \((S_i)\) holds for any \(i \geq 3.\)

Let \(k \leq (m + 1)/2\) and assume that \((S_{2k-1})\) does not hold, i.e., that there exist elements \(x^i_j, y^i_j\) such that \(x^i_j \neq x^{i+1}_j\) for all \(i \leq k < p\) and either \(y^i_j = x^i_j\) or \(y^i_j = x^i_j + k,\)

and such that

\[
\begin{align*}
(x_1, x^1_2, \ldots, x^n_1) &\succ (x^1_1, y^1_2, \ldots, y^n_1), \\
(x_1, x^1_2, \ldots, x^n_1) &\succ (x^{i-1}_1, y^i_2, \ldots, y^n_1), \ i \in \{2, \ldots, k\}, \\
(x^i_1, x^{i+1}_2, \ldots, x^{i+1}_k) &\succ (x^{i+1}_1, y^{i+1}_2, \ldots, y^{i+1}_k), \\
(x^{i+j}_1, x^{i+j}_2, \ldots, x^{i+j}_k) &\succ (x^{i+j-1}_1, y^{i+j}, \ldots, y^{i+j}_k), \ j \in \{2, \ldots, k\},
\end{align*}
\]

with at least one strict preference relation \(\succ\). If there exists no \(j\) such that \(x^j_2 = 2m\) or \(y^j_2 = 2m\), then (18) cannot occur because it would violate a cancellation axiom and because \(\succ\) is represented on this subspace by an additive utility function. So, suppose that there exists at least one index \(j \in \{1, \ldots, 2k\}\) such that \(x^j_2 = 2m\) or \(y^j_2 = 2m\).

When \(x^j_2 = y^j_2,\) by independence substitute \(x^j_2\) and \(y^j_2\) by 0’s. Now, by symmetry, one can suppose that there exist \(p\) indices \(\{i_1, i_2, \ldots, i_p\}, \ i_j \leq k,\) such that \(y^{i_j}_2 = 2m.\) Using transformation (12) as in the proof of theorem 2, we can furthermore suppose that \(i_1, \ldots, i_p\) are the first \(p\) indices. This transformation has no consequence on the rest of the proof except to simplify notation. Now translating the \(k\) first preference relations into their \(U\) counterpart, we get:

\[
\sum_{i=1}^{k} \sum_{j=1}^{n} x^i_j + kH \geq \sum_{i=1}^{p-1} V(x^i_1) + V(x^i_1) + \sum_{i=p+1}^{k} (x^{i-1}_1 + y^{i}_2 + H) + \sum_{i=1}^{k} \sum_{j=3}^{n} x^i_j.
\]

Thus,

\[
\sum_{i=1}^{k} x^i_2 \geq \sum_{i=1}^{p-1} [V(x^i_1) - x^i_1] + [V(x^i_1) - x^i_1] + \sum_{i=p+1}^{k} y^{i}_2 - pH + \sum_{i=1}^{k} \sum_{j=3}^{n} (y^{i}_j - x^{i}_j).
\]

Now by (5), \(V(x) \geq x - 0.5,\) so the inequality above implies that

\[
\sum_{i=1}^{k} x^i_2 \geq -0.5p + \sum_{i=p+1}^{k} y^{i}_2 - pH - k(n - 2) \geq -0.5p - pH - k(n - 2).
\]

But \(k \leq \frac{m+1}{2},\) so, expanding \(H,\) we get

\[
\sum_{i=1}^{k} x^i_2 \geq -0.5p + p[m^2 + mn + 3n - 8.5] - \frac{m+1}{2}(n - 2)
\]

\[
\geq pm^2 + \left(p - \frac{1}{2}\right) mn + m + \left(3p - \frac{1}{2}\right)n - 9p + 1.
\]

Since \(m \geq 3, n \geq 2\) and \(p \geq 1,\)

\[
\sum_{i=1}^{k} x^i_2 \geq pm^2 + 6 \left(p - \frac{1}{2}\right) + 3 + 2 \left(3p - \frac{1}{2}\right) - 9p + 1 \geq pm^2 + 3p \geq m^2 + 3.
\]
But this is impossible because $k \leq \frac{m+1}{2}$ and $x^i_2 \leq 2(m-1)$ since $x^i_2 \neq y^i_j = 2m$, so that

$$\sum_{i=1}^{k} x^i_2 \leq \frac{m + 1}{2} \cdot 2(m - 1) = m^2 - 1.$$ 

Hence (18) cannot hold. Thus, $(S_i)$ holds for any $i \in \{3, \ldots, m\}$, and consequently $(C_m)$ also holds. Since $m$ is odd, by theorem 2, $(C_{m+1})$ also holds. 

In order to prove lemma 6, I introduce the following lemma:

**Lemma 7**: Let $\succsim$ be a weak order on $X = X_1 \times X_2$. Assume restricted solvability w.r.t. $X_1$ (axiom 6), independence (axiom 1) and the Archimedean axiom (axiom 12). Suppose that $\succsim$ is representable by a utility function, and that $\text{Card}(X_2) = 2$. Then $\succsim$ is representable by an additive utility function.

**Proof of lemma 7**: Let $X_2 = \{a, b\}$. If $a \sim b$, the proof is obvious. Assume henceforth that $b \succ a$. Let $\tilde{X}_1$ be the set of equivalent classes of $\succsim_1$. The existence of an additive utility function on $X_1 \times \{a, b\}$ is equivalent to the existence of an additive utility function on $\tilde{X}_1 \times \{a, b\}$. The advantage of working in the latter set rather than in $X_1 \times \{a, b\}$ is that the utility function is one-to-one in this space. So, let us work in $\tilde{X}_1 \times \{a, b\}$.

By hypothesis, $\succsim$ is representable by a utility function, say $u(\cdot, \cdot)$. Let $u_a(\cdot) = u(\cdot, a)$ and $u_b(\cdot) = u(\cdot, b)$. To prove the existence of an additive utility, it is sufficient to prove that there exists a strictly increasing function, say $\varphi(\cdot)$, transforming $u(\cdot)$ into an additive function. In other words, there exists $\alpha \in \mathbb{R}^+$ such that

$$\varphi \circ u_b(x) = \varphi \circ u_a(x) + \alpha, \text{ for all } x \in \tilde{X}_1. \tag{19}$$

Suppose that $(x_1, a) \prec (y_1, b)$ for all $x_1, y_1 \in \tilde{X}_1$. Then, in terms of utility functions, $u_a(\tilde{X}_1) \cap u_b(\tilde{X}_1) = \emptyset$. Let $\varphi(\cdot)$ be defined by:

$$\varphi(x) = \begin{cases} \arctan(x) & \text{if } x \in u_a(\tilde{X}_1), \\ \arctan(u_a(u_b^{-1}(x))) + \pi & \text{if } x \in u_b(\tilde{X}_1). \end{cases}$$

Then it is obvious that $\varphi \circ u(\cdot, \cdot)$ is an additive utility representing $\succsim$ on $\tilde{X}_1 \times \{a, b\}$.

Now, suppose that $u_a(\tilde{X}_1) \cap u_b(\tilde{X}_1) \neq \emptyset$. (19) is equivalent to the following equation:

$$\varphi \circ (u_b \circ u_a^{-1})(x) = \varphi(x) + \alpha, \text{ for all } x \in u_a(\tilde{X}_1). \tag{20}$$

Note that the value of $\alpha$ is not important, as long as it remains strictly positive. Indeed, if it were replaced by $\beta > 0$, $\varphi(\cdot)$ would be replaced by $\frac{\beta}{\alpha} \varphi(\cdot)$. Let $x^0 \in u_a(\tilde{X}_1)$. The value of $\varphi(x^0)$ can be taken arbitrarily; indeed, adding a constant to $\varphi(\cdot)$ has no consequence on the validity of (20). Now, let us construct a function $\varphi(\cdot)$ satisfying (20).

We first define $\varphi(\cdot)$ on $[x^0, u_b \circ u_a^{-1}(x^0)]$ as:

$$\varphi(x) = \frac{\alpha(x - x^0)}{u_b \circ u_a^{-1}(x^0) - x^0} + \varphi(x^0), \text{ for all } x \in [x^0, u_b \circ u_a^{-1}(x^0)]. \tag{21}$$
Note that since \((x, a) < (x, b)\) for every \(x \in \tilde{X}_1\), \(u_b \circ u_{a}^{-1}(x^0) > x^0\), so that \(\varphi(\cdot)\), as defined by equation (21), is well defined and is strictly increasing.

Extend \(\varphi(\cdot)\) as follows: for all \(x \in [x^0, u_b \circ u_{a}^{-1}(x^0)] \cap u_a(\tilde{X}_1)\), equation (20) can be applied. Let \(x^1 = \sup\{x : x \in [x^0, u_b \circ u_{a}^{-1}(x^0)] \cap u_a(\tilde{X}_1)\}\). Then, applying (20) for all \(x \in [x^0, x^1[ \cap u_a(\tilde{X}_1)\) if \(x^1\) is not reached (resp. \([x^0, x^1] \cap u_a(\tilde{X}_1)\) otherwise), we define \(\varphi(\cdot)\) on \([u_b \circ u_{a}^{-1}(x^0), u_b \circ u_{a}^{-1}(x^1)]\) \(\cap u_b(\tilde{X}_1)\) (resp. \([u_b \circ u_{a}^{-1}(x^0), u_b \circ u_{a}^{-1}(x^1)]\) \(\cap u_b(\tilde{X}_1)\)). In fact, restricted solvability w.r.t. \(\tilde{X}_1\) ensures that we define \(\varphi(\cdot)\) on \([u_b \circ u_{a}^{-1}(x^0), u_b \circ u_{a}^{-1}(x^1)]\) \(\cap u_b(\tilde{X}_1)\) (resp. \([u_b \circ u_{a}^{-1}(x^0), u_b \circ u_{a}^{-1}(x^1)]\) \(\cap (u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1))\)). Moreover, \(\varphi(\cdot)\) strictly increases because, by independence, \(u_b \circ u_{a}^{-1}(x) > x\) for all \(x \in u_a(\tilde{X}_1)\) and \(\alpha > 0\).

Now, two cases can occur. In the first one, \(x^1 = u_b \circ u_{a}^{-1}(x^0)\) but \(x^1 \notin u_a(\tilde{X}_1)\), or \(x^1 < u_b \circ u_{a}^{-1}(x^0)\). Then, by restricted solvability, there exists no element, say \(y \in \tilde{X}_1\), such that \(u_a(y) > x^1\) — otherwise, by definition of \(x^1\), \(u_a(y) > u_b \circ u_{a}^{-1}(x^0)\), which would imply that \((y, a) \succ (u_{a}^{-1}(x^0), b) \succ (u_{a}^{-1}(x^0), a)\) and would ensure the existence of \(t \in \tilde{X}_1\) such that \((t, a) \sim (u_{a}^{-1}(x^0), b)\). Therefore, \(\varphi(\cdot)\) has been defined on \([x \in u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1)\) such that \(x \geq x^0\).

In the second case, \(x^1 = u_b \circ u_{a}^{-1}(x^0)\) and \(x^1 \in u_a(\tilde{X}_1)\). Then \(\varphi(\cdot)\) has been defined on \([a, x^1]\) \(\cap (u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1))\). Let \(x^2 = \sup\{x : x \in [x^1, (u_b \circ u_{a}^{-1})^2(x^0)] \cap u_a(\tilde{X}_1)\}\). A process similar to that of the preceding paragraphs can be applied with the set \([x^1, x^2]\) instead of \([x^0, x^1]\). By continuing the same process, we construct a sequence of points \((x^n)_{n=0}^\infty\), where \(n\) is either an integer or the infinity. \(\varphi(\cdot)\) is then defined either on \([x^0, x^n]\) \(\cap (u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1))\) or on \([x^0, x^n]\) \(\cap (u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1))\), depending on whether \(x^n\) is reached or not. If \(n\) is finite, by restricted solvability w.r.t. the first component, \(\varphi(\cdot)\) has been defined on the set \([x \in u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1) : x \geq x^0]\).

Problems arise when \(n\) is infinite. In such a case, \(\varphi(\cdot)\) has been defined on \([x^0, (u_b \circ u_{a}^{-1})^n(x^0)] \cap (u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1))\). Now, suppose that there exists \(x \in u_b(\tilde{X}_1)\) such that, for every integer \(i\), \((u_b \circ u_{a}^{-1})^i(x^0) < x\). By construction, \(\varphi((u_b \circ u_{a}^{-1})^i(x^0)) = \varphi(x^0) + \alpha i\); hence, this should tend toward \(+\infty\) when \(i\) tends toward \(+\infty\). But, then, the problem is that \(\varphi(x)\) cannot be finite, and so \(\varphi(\cdot)\) is not well defined on \(u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1)\).

Fortunately, this case cannot arise. For all integers \(i, i+1 \geq 0\), \(x^{i+1} = u_b \circ u_{a}^{-1}(x^i)\). Now, by the process of construction, \(x^i, x^{i+1} \in u_b(\tilde{X}_1)\); so, there exists a sequence of elements of \(\tilde{X}_1\), say \((y^i)_{i \geq 0}\), such that \(x^i = u_a(y^i)\) for all \(i\). But, then, \(u_a(y^{i+1}) = u_b(y^i)\). In terms of preference relations, \((y^{i+1}, a) \sim (y^i, b)\). Therefore, \((y^i)\) is a standard sequence. So, by the Archimedean axiom, if \(x\) existed such that \(x\) is greater than \((u_b \circ u_{a}^{-1})^i(x^0)\) for all \(i\), then \(\{i\}\) would be finite. Hence, the case described in the previous paragraph would not arise. Consequently, \(\varphi(\cdot)\) has been well defined on \(\{x \in u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1) : x \geq x^0\}\).

Now, using (20) conversely, i.e.,

\[
\varphi \circ (u_a \circ u_{a}^{-1})(x) = \varphi(x) - \alpha, \text{ for all } x \in u_b(\tilde{X}_1),
\]

and applying a process similar to that of the preceding paragraphs, starting with the set \([x^0, (u_b \circ u_{a}^{-1})(x^0)]\), one can construct \(\varphi(\cdot)\) on \(\{x \in u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1) : x^0 \geq x\}\). And, of course, as shown above, \(\varphi(\cdot)\) is strictly increasing. Hence, \(\varphi(\cdot)\) exists on \(u_a(\tilde{X}_1) \cup u_b(\tilde{X}_1)\).

Note that \(\varphi(\cdot)\) is not unique up to positive affine transformations because \(\varphi(\cdot)\) can be defined with a certain degree of freedom in equation (21).
Proof of lemma 6: Suppose that a utility function exists, representing $\succeq$ on $X$. Consider arbitrary elements $i, i+1 \in N$. By lemma 7, there exists an additive utility, say $u$, representing $\succeq$ on $X_1 \times \{x_{i}^{j}, x_{i+1}^{j}\}$. This means that there exists a constant, say $\alpha$, such that $u(\cdot, x_{i+1}^{j}) = u(\cdot, x_{i}^{j}) + \alpha$. Suppose that $i + 2 \in N$. Let $A = \{y \in X_1 : \text{there exists } x \in X_1 \text{ such that } (y, x_{i+1}^{j}) \sim (x, x_{i}^{j})\}$. (6) defines uniquely $u(\cdot, x_{i+2}^{j})$ on $A$ because $(x, x_{i+1}^{j}) \sim (y, x_{i+1}^{j}) \iff u(x, x_{i+1}^{j}) = u(y, x_{i+1}^{j}) = u(y, x_{i}^{j}) + \alpha \iff (y, x_{i+2}^{j}) = u(x, x_{i+1}^{j}) = u(x, x_{i}^{j}) + \alpha = u(y, x_{i}^{j}) + 2\alpha$. Obviously, $u$, as defined above, is a utility function over $A \times \{x_{i}^{j}, x_{i+1}^{j}, x_{i+2}^{j}\}$.

Now, there remains to extend $u(\cdot, x_{i+2}^{j})$ on $B = \{y \in X_1 : \text{there exists } x \in X_1 \text{ such that } (y, x_{i+1}^{j}) \sim (x, x_{i}^{j})\}$. But, by restricted solvability, $B = \{y \in X_1 : (y, x_{i+1}^{j}) \succ (x, x_{i}^{j}) \text{ for all } x \in X_1\}$ and, by definition 2, $B = \{y \in X_1 : (y, x_{i+2}^{j}) \succ (x, x_{i+1}^{j}) \text{ for all } x \in X_1\}$. So, $u(\cdot, x_{i+2}^{j})$ defined on $B$ as $u(\cdot, x_{i+2}^{j}) + \alpha$ fits as a utility function. Indeed, let

$$\beta = \sup_{x \in B} u(x, x_{i}^{j}) - \inf_{y \in B} u(y, x_{i}^{j}).$$

Then $\beta < \alpha$ if both the sup and the inf are reached, else $\beta \leq \alpha$. If this were not true, there would exists $(x, y) \in B \times B$ such that $u(x, x_{i}^{j}) - u(y, x_{i}^{j}) \geq \alpha$, or equivalently such that $(x, y) \succeq (y, x_{i}^{j})$. But this leads to a contradiction because $y \in B$. Since $u$ is additive on $X_1 \times \{x_{i}^{j}, x_{i+1}^{j}\}$, $\beta = \sup_{x \in B} u(x, x_{i+1}^{j}) - \inf_{y \in B} u(y, x_{i+1}^{j})$. Hence $u(x, x_{i+2}^{j}) = u(x, x_{i+1}^{j}) + \alpha$ for all $x \in B$ guarantees that $(x, x_{i+2}^{j}) \succ (y, x_{i+1}^{j})$ for all $x, y \in B \times X_1$. Moreover, $u(x, x_{i+2}^{j}) \geq u(y, x_{i+2}^{j}) \iff u(x, x_{i+1}^{j}) \geq u(y, x_{i+1}^{j})$ for all $x, y \in B$, so $u$, as defined above, preserves the ordering.

Hence, there exists an additive utility function on $X_1 \times \{x_{i}^{j} : j \in N \text{ and } j \geq i\}$. By induction, we construct the utility function on $X_1 \times \{x_{j}^{j} : j \in N \text{ and } j \geq i\}$. With a similar process, we construct $u$ on $X_1 \times \{x_{j}^{j} : j \in N \text{ and } j \leq i\}$. So, an additive utility function exists, representing $\succeq$ on $X$. ■

References


Fishburn, P. C.: 1997a, Cancellation conditions for finite two-dimensional additive measurement, communicated by the author.


