Tile Packing Tomography is NP-hard

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Tables in an open space office

- disjoint copies of the same shape
- same orientation
Non-intrusive measurement

- the office is an \( n \times m \) grid
- tables are aligned on the grid
- Measurement results in projection vectors \( r, s \)
- such that \( r_i \) is the number of grid cells of row \( i \) covered by a table (tile)
- same for columns
Equivalent measurement

- alternative measurement (equivalent up to base change):
  - mark a cell in the tile
  - projections count only marks
• A tile is a connected set of grid points

• Given a tile T, dimensions n,m and projections r, s

• does there exist a binary matrix M

• with \( r_i = \sum_j M_{ij} \), \( s_j = \sum_i M_{ij} \)

• and for \( M_{ij} = 1 \), \( M_{i'j'} = 1 \), the tiles \( T+(i,j) \) and \( T+(i',j') \) are disjoint?
A tile is a connected set of grid points.

Given a tile \( T \), dimensions \( n,m \) and projections \( r, s \),

does there exist a binary matrix \( M \) with

\[
\begin{align*}
    r_i & = \sum_j M_{ij}, \\
    s_j & = \sum_i M_{ij}
\end{align*}
\]

and for \( M_{ij} = 1 \), \( M_{i'j'} = 1 \), the tiles \( T+(i,j) \) and \( T+(i',j') \) are disjoint?
The tiling reconstruction pb

- A tile is a connected set of grid points.
- Given a tile $T$, dimensions $n,m$ and projections $r, s$
- does there exist a binary matrix $M$
- with $r_i = \sum_j M_{ij}$, $s_j = \sum_i M_{ij}$
- and for $M_{ij} = 1$, $M_{i'j'} = 1$, the tiles $T^+(i,j)$ and $T^+(i',j')$ are disjoint?
Complexity depends on $T$

- When the tile $T$ is a bar, the problem is polynomial.
- [this paper] When the tile $T$ is not a bar, the problem is NP-hard.

- [Ryser’63] Characterize $r,c$ such that there is a binary matrix with projections $r,c$.
- [Picouleau’01] [D, Gol, Rap, Rémi a’03] greedy algorithm to reconstruct tilings with bars.
- [Chrobak, Couperous, D, Woeginger’03] NP-hardness for some very specific tiles.
3-color tomography

- 3 colors \{R,G,B\}
- given projections \(r^c, s^c\) for every \(c \in \{R,G,B\}\)
- is there a matrix \(M \in \{R,G,B\}^{n \times m}\)
- such that \(r^c_i = \# \{j : M_{ij} = c\}\)
- and \(s^c_j = \# \{i : M_{ij} = c\}\)
- for every \(c \in \{R,G,B\}\)
- \([D, Guíñez, Matamala’09]\) 3-color tomography is NP-hard
Reduction from 3-color tomography

- Reduce from 3-color tomography to tiling tomography

- Choose a block of fixed dimension $k \times l$

- Choose 3 tilings of the block

\[ \begin{align*}
M^R & = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \\
M^G & = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \\
M^B & = \begin{bmatrix}
2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{bmatrix}
\end{align*} \]
Reduction from 3-color tomography

- **[in]** instance \( r^c, s^c \) (\( c \in \{R,G,B\} \)) of the 3-color tomography problem for an \( n \times m \) grid

- **[out]** instance \( r, s \) of the tiling tomography problem for an \( nk \times ml \) grid such that projections of block row \( i \) are

\[
r_i^R \cdot r_i^R + r_i^G \cdot r_i^G + r_i^Y \cdot r_i^Y
\]

(same for columns)
Requirements

• (R1) the row projections $r^R, r^G, r^Y$ have to be affine linear independent

• (R2) Let $M$ be a solution to the tiling tomography instance obtained by the reduction. Then every block in $M$ is one of $M^R, M^G, M^Y$ (or projection-equivalent)
Implications

- (R2) ⇒ we can associate a color to every block in M

- and replace every block by a single colored cell (contract)

- (R1) ⇒ the obtained grid has the required projections, since any vector
  \[ n_R \cdot r^R + n_G \cdot r^G + n_Y \cdot r^Y \]
  for \( n_R + n_G + n_Y = n \)
  is uniquely decomposed into \( n_R, n_G, n_Y \).
Apply this technique

- We divide the tiles into four classes
- and have a different construction for every class
- Fix a maximal conflicting vector \((p,q)\)
- Choose smallest \(a>0\) such that \((ap,0)\) is not conflicting
- Choose smallest \(b>0\) such that \((0,bq)\) is not conflicting
- Cases are broken according to \(a,b,p,q\)
Example case $b=1$, $a \geq 2$

- We choose $k,l$ large enough
- block tilings are as depicted, (R1) ok
We have to show:

(R2) Let $M$ be a solution to the tiling tomography instance obtained by the reduction. Then every block in $M$ is one of $M^R, M^G, M^Y$ (or projection-equivalent).

There might be another block tiling in the solution, namely $M^A$.

It counts like $M^R$ in the column projections and like $M^G$ in the row projections.

Since total row projections equal total column projections this is impossible.
• What about approximation algorithms?

• What about complete tilings, for a constant number of tiles?