Finding total unimodularity

in optimization problems solved by linear programs

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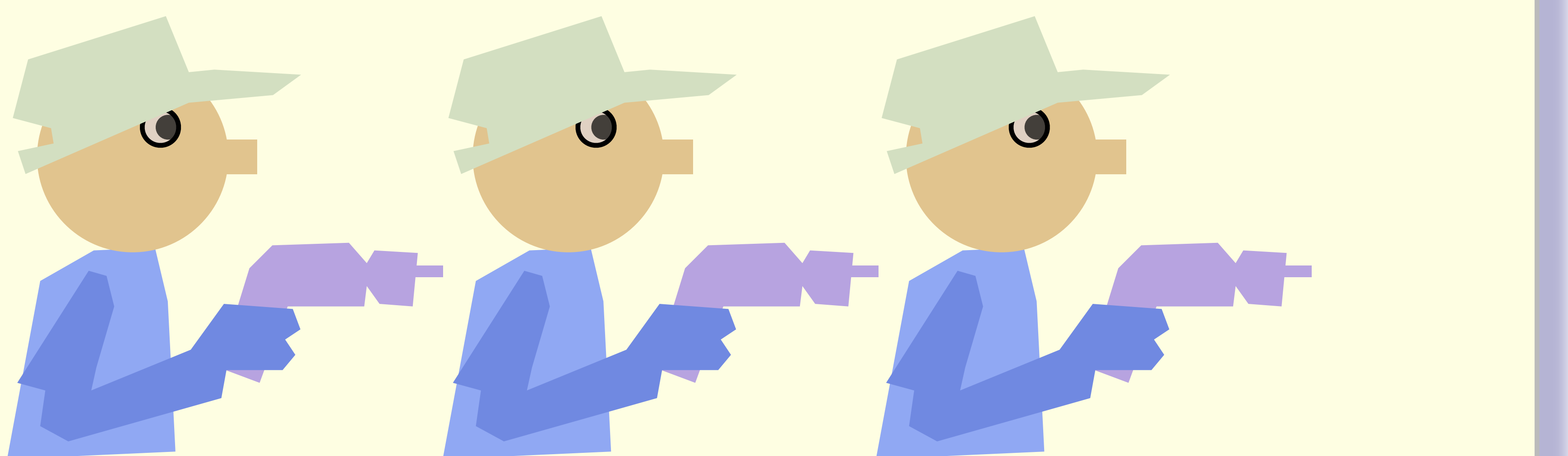
mapsp istanbul july 07

The motivating story

THE IKEA KITCHEN BUILDERS

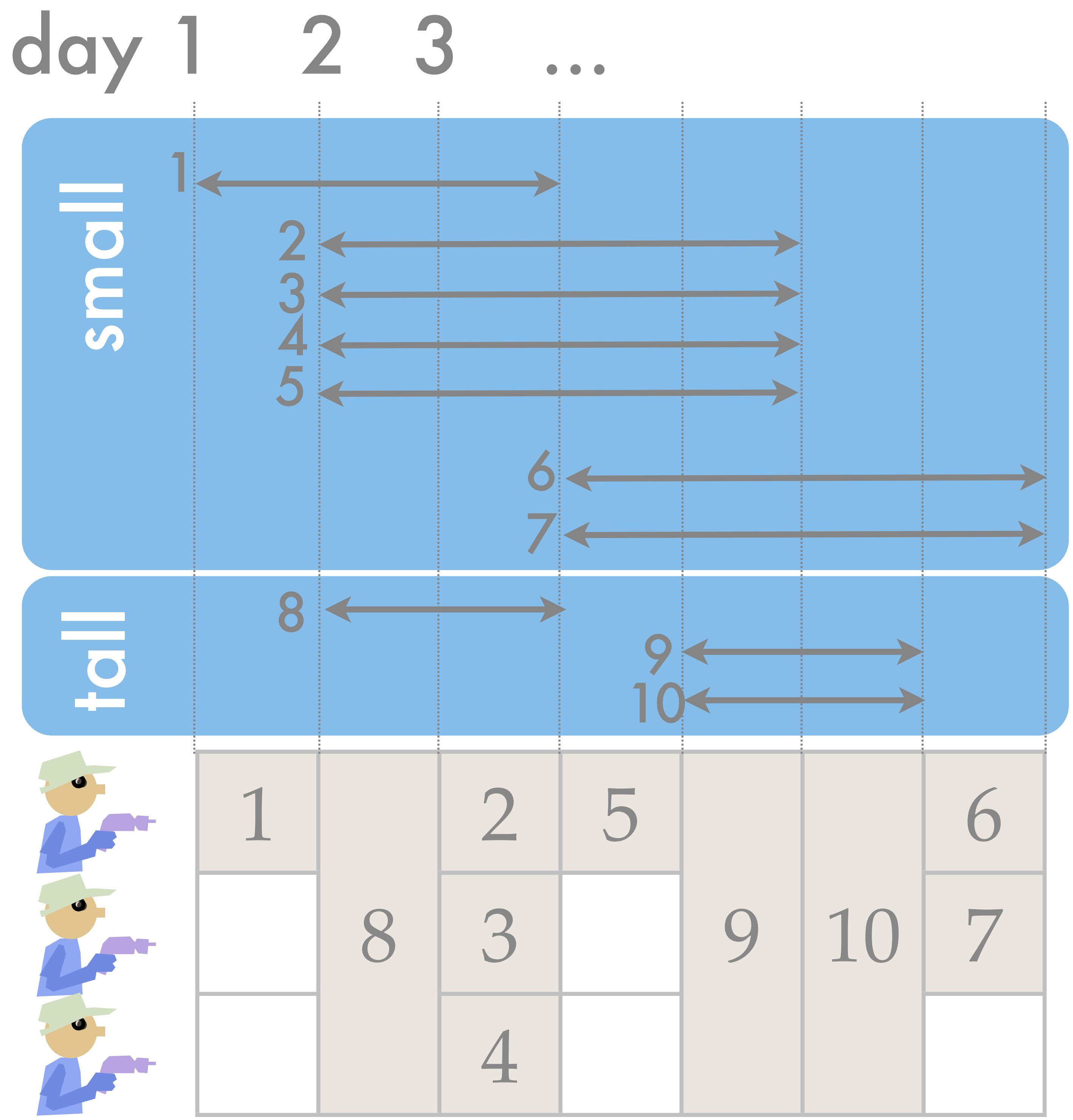
We build your kitchen in one day



Just call 0 800 222 333
and tell us two dates between which
you wish us to come

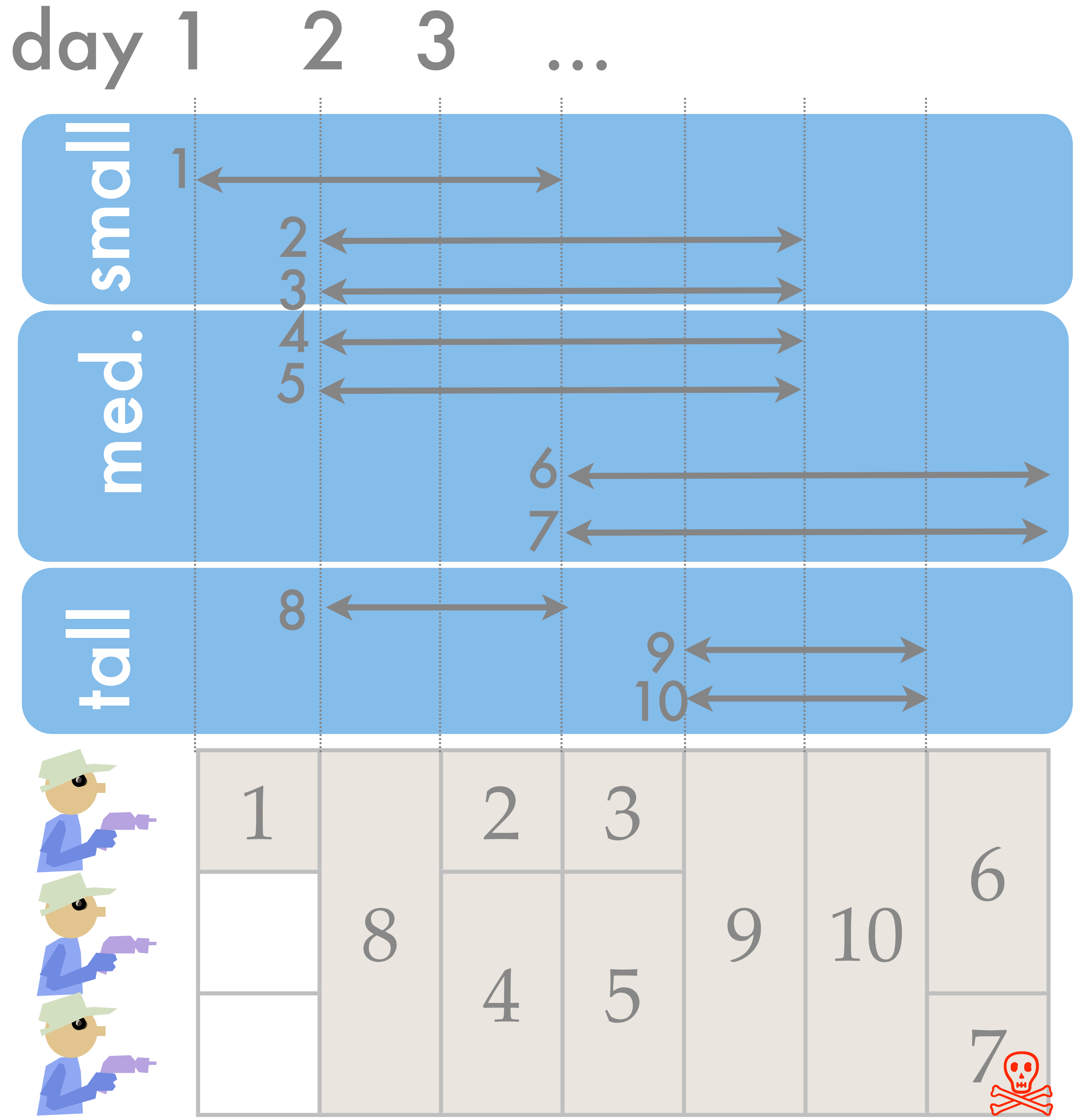
A scheduling problem

- Orders are n jobs with release times and strict deadlines (integers)
- Processing time =1
- We have 3 parallel machines
- However some jobs require only 1 arbitrary machine, while some require all 3 **(small/tall jobs)**
- Goal: find feasible schedule serving all jobs

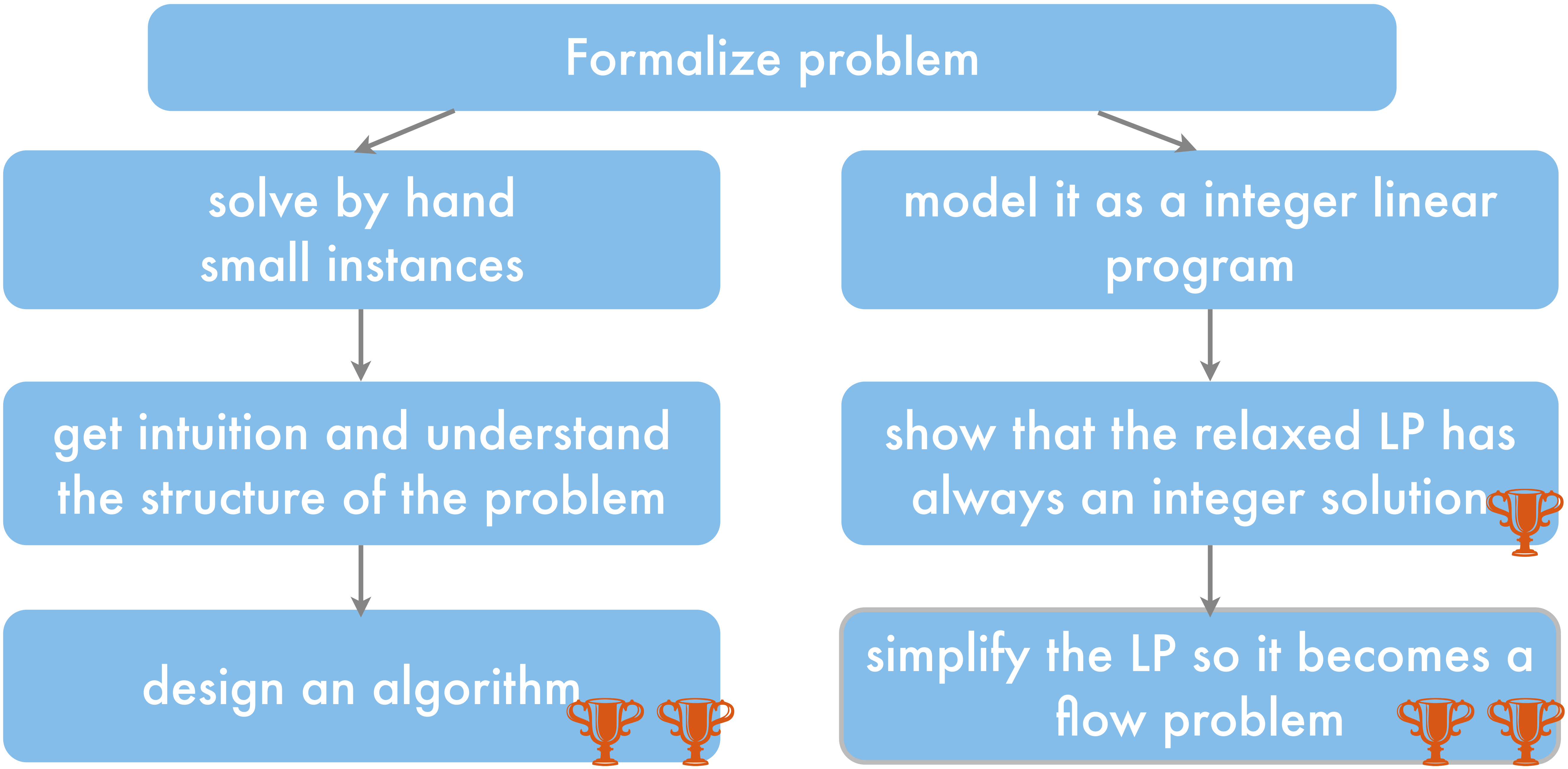


(Open problem)

- 3 parallel machines
- n jobs with release time r_j, deadline d_j, and arbitrary size $\mathbf{s_j \in \{1,2,3\}}$
- Is is possible to find a feasible schedule in polynomial time ?



How to find a polynomial time algorithm ?



Model it as an integer LP

- First : we can assume that there are n time points (1..n) otherwise the problem consists of independent subproblems.
- Each job j has a release time r_j, deadline d_i, size $s_j \in \{1,3\}$
- $X_{jt}=1$ means job j schedules at time t (else $X_{jt}=0$)
- solve (no objective function)

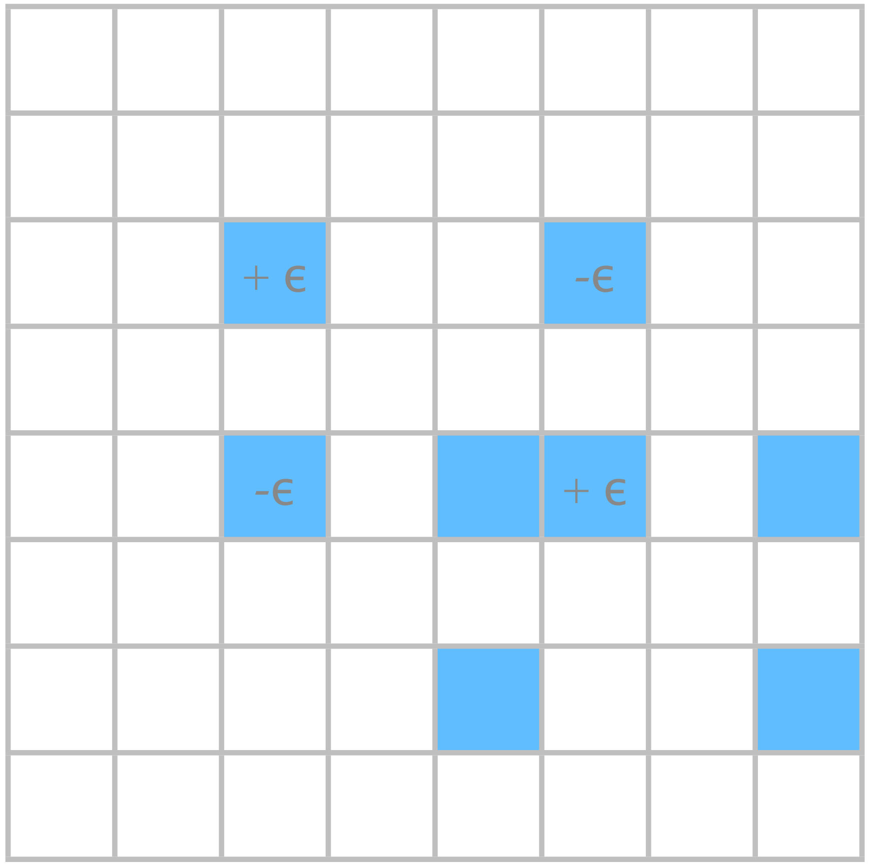
$\forall j \forall t$	$0 \le X_{jt} \le 1$	
$\forall j \forall t \notin [r_j,d_j)$	$X_{jt}=0$	– release times, deadlines are respected
$\forall j$	$\sum t\ X_{jt}=1$	– every job is scheduled
$\forall t$	$\sum j\ s_j X_{jt} \le 3$	– 3 machines are enough

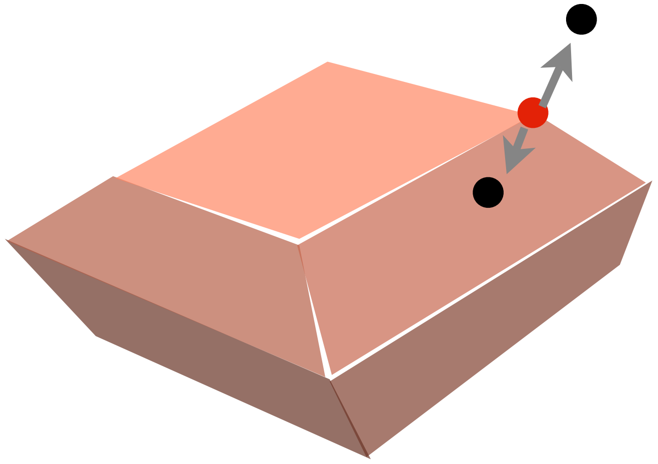
Integral solutions

- Theorem [Baptiste,Schieber'03] If the LP is feasible then it has an integral solution, in fact they give some objective function and show that the optimum is integral
- The LP has $O(n^2)$ variables and $O(n^2)$ constraints
- So the problem can be solved in expected time $O(n^4)$ and worst case time $O(n^{10})$

General proof technique

- Fix some objective function, and some fractional solution X
- Show that (1) either it is not optimal, or (2) it is not a vertex of the polytope
- (2:) Find two disjoint sets of fractional variables S^-, S^+ such that every tight constraint contains as many var. from S^- than from S^+
- Then decreasing S^- by ϵ and increasing S^+ or the other way round preserves all constraints. So this is not a vertex.





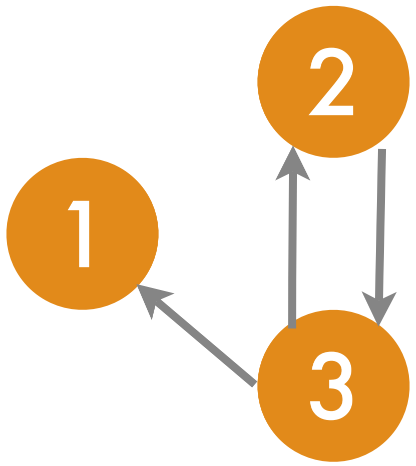
When does a LP has integral solution ?

- min $c^T x$, s.t. $Ax \leq b$, b integral
- Theorem: all vertices of the polytope are integral if A is totally unimodular, i.e. for any c,b, the LP has an integral optimum
- Defintion: A is *totally unimodular* if every squared submatrix has determinant -1,0, or +1

Total unimodularity

- Thm [Poincaré'1900] The incidence matrix of a directed graph is totally unimodular

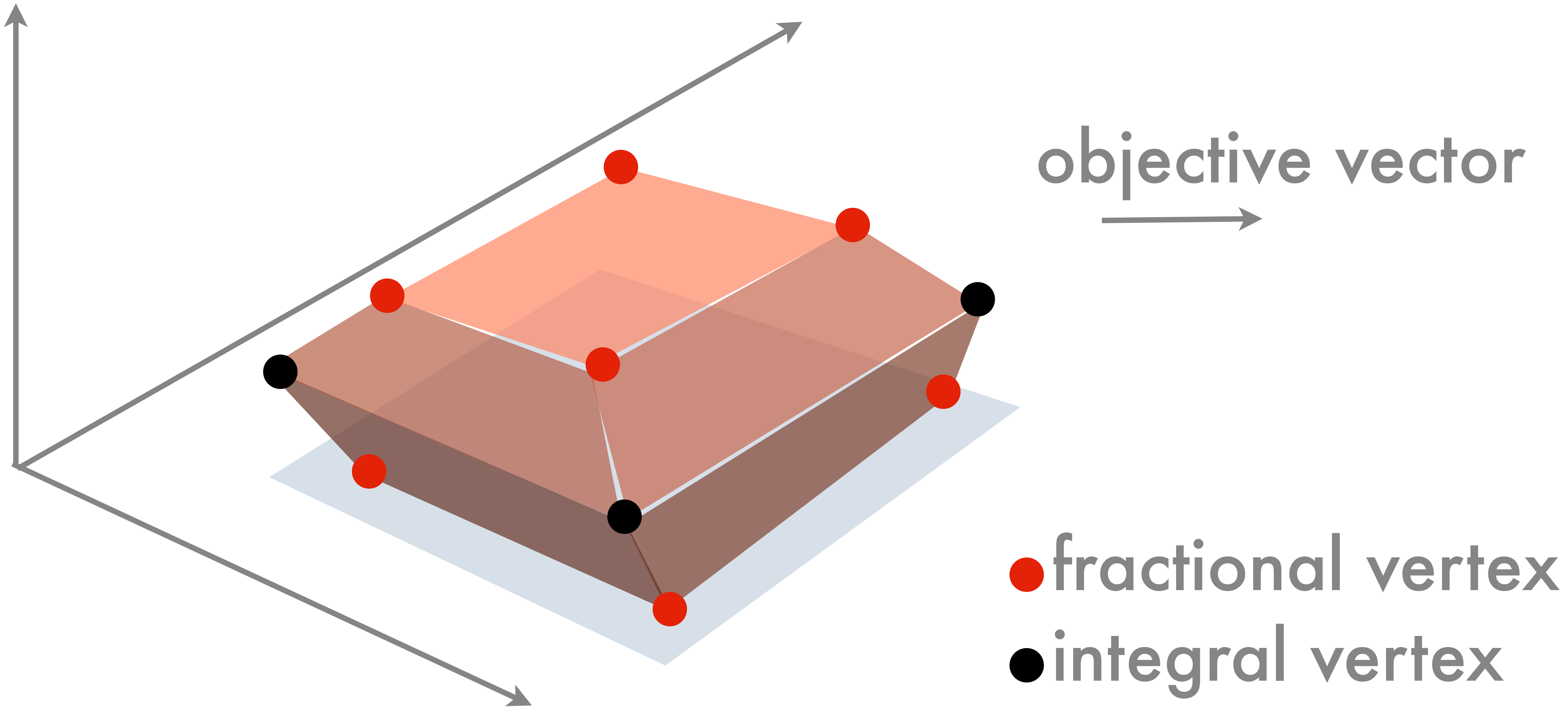
$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



- Thm [Seymour'83] The set of all totally unimodular matrices is generated by all incidence matrices, and two specific 5x5 matrices under some operations.

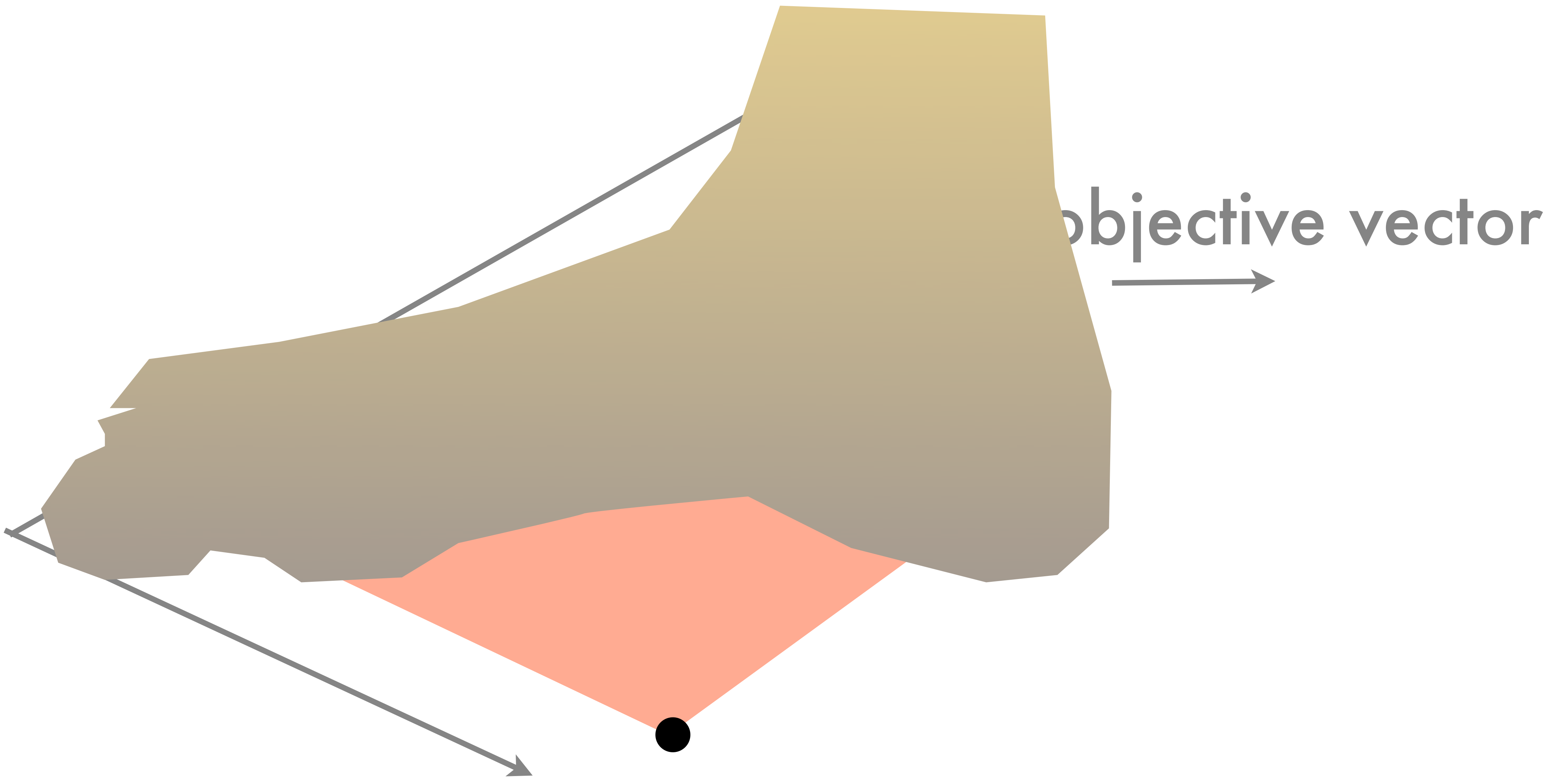
Our contribution

- Baptiste and Schieber's LP is not totally unimodular (it has coefficients 3), but still has always an integral optimum



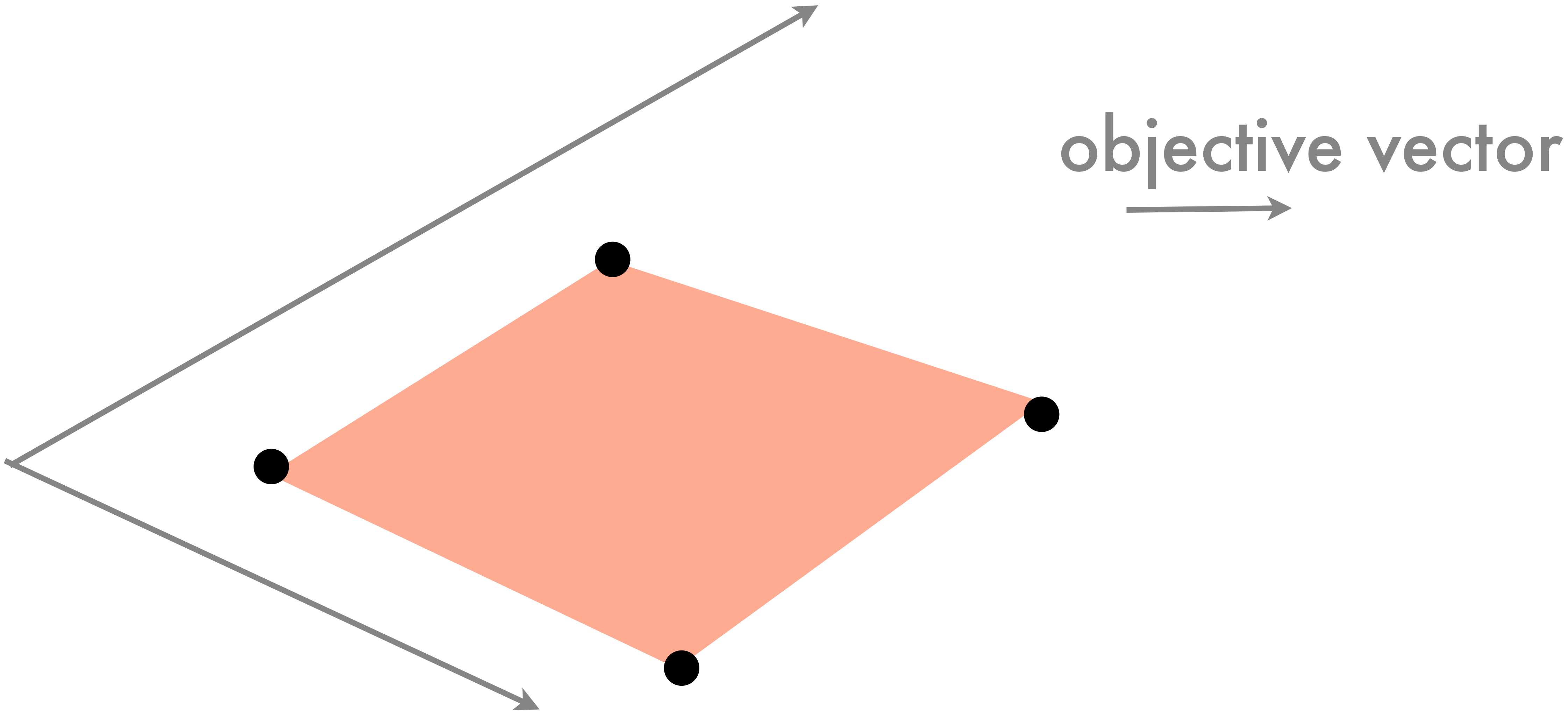
Our contribution

- We project the search space on a lower dimensional space



Our contribution

- The result is a LP which is totally unimodular and corresponds to a shortest path problem



Our new linear program

X_{jt}

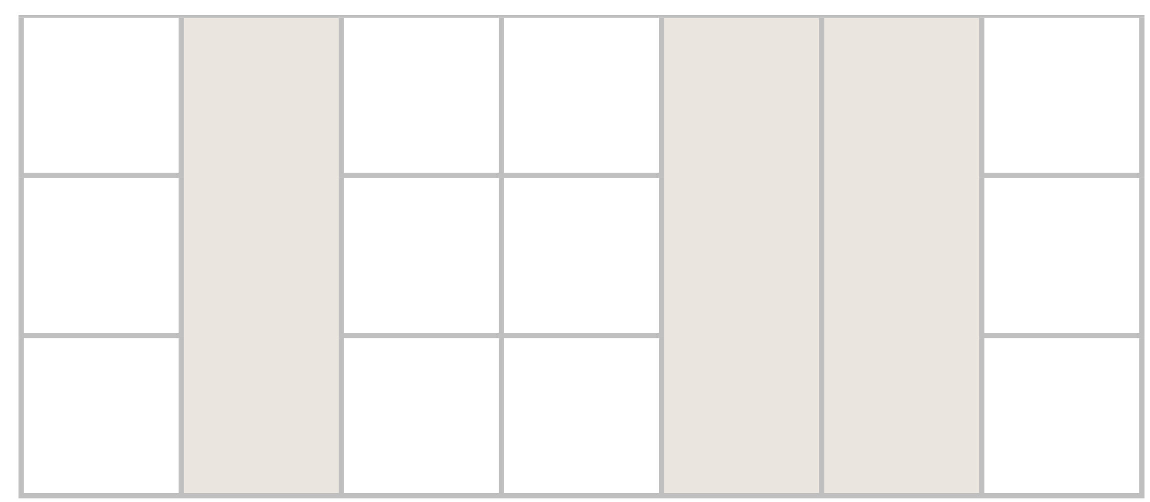
$i \downarrow$

1	0	0	0	0	0	0
0	0	1	0	0	0	0
0	0	1	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	1
0	0	0	0	0	0	1
0	1	0	0	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0

- Instead of computing the actual schedule (X_{jt}) we only decide on the execution slots for large jobs (Y_t=1 if there is a large job scheduled at t)

Y_t

0	1	0	0	1	1	0

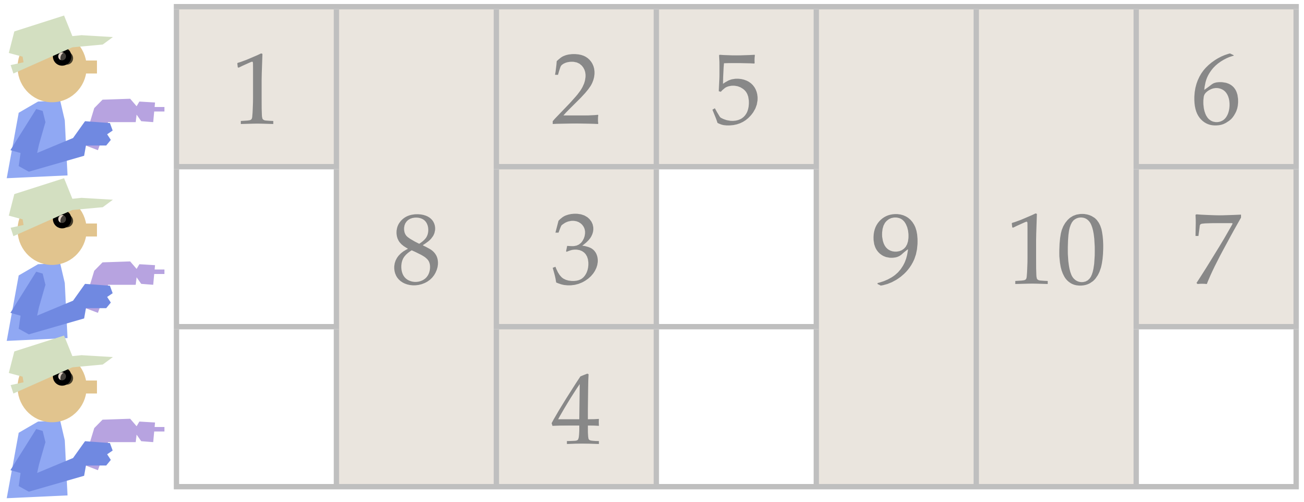


Our new linear program

X_{jt}

i ↓

1	0	0	0	0	0	0
0	0	1	0	0	0	0
0	0	1	0	0	0	0
0	0	1	0	0	0	0
0	0	0	1	0	0	0
0	0	0	0	0	0	1
0	0	0	0	0	0	1
0	1	0	0	0	0	0
0	0	0	0	1	0	0
0	0	0	0	0	1	0



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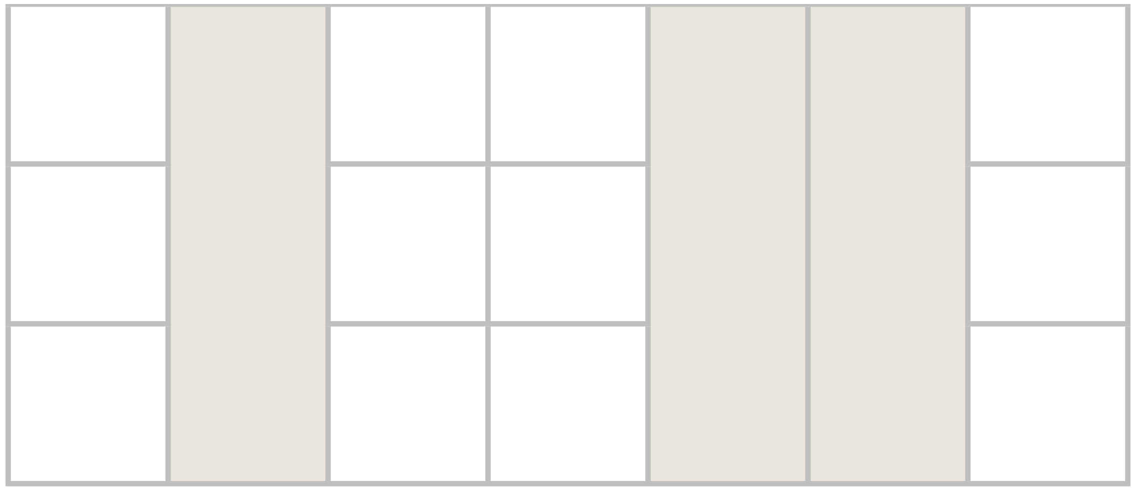

Our new linear program

- Sum constraints on X_{jt} to get constraints on Y_t

- $\forall j \forall t$ $0 \leq X_{jt} \leq 1$

  $\forall j \forall t \notin [r_j, d_j)$ $X_{jt} = 0$

  $\forall j$ $\sum t \; X_{jt} = 1$

  $\forall t$ $\sum j \; s_j X_{jt} \leq 3$

Our new linear program

- Sum constraints on X_{jt} to get constraints on Y_t

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  $\forall j$ $\quad \sum t\, X_{jt} = 1$

  $\forall t$ $\quad \sum j\, s_j X_{jt} \le 3$

- $\forall t$ $\quad 0 \le Y_t \le 1$

  $\forall s,u$ $\quad \sum t \in [s,u)\ Y_t \ge A_{su}$

  $\forall s,u$ $\quad u-s-\sum t \in [s,u)\ Y_t \ge \lceil B_{su}/3 \rceil$

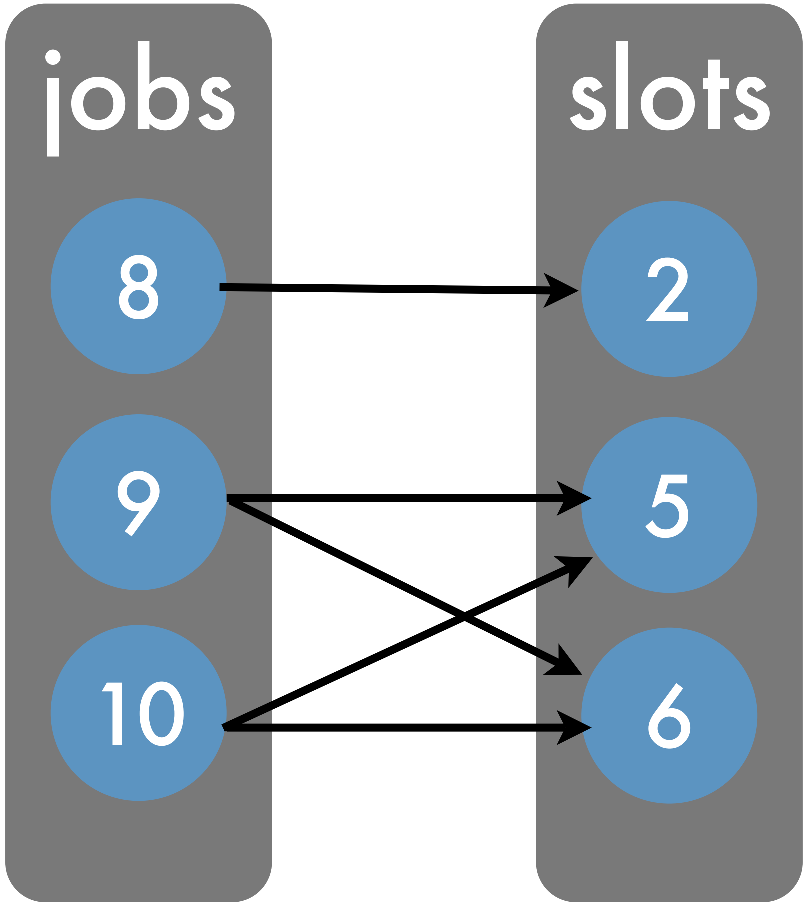
- where

  $A_{su} := |\{j: s_j=3 \wedge [r_j,d_j] \subseteq [s,u]\}|$

  $B_{su} := |\{j: s_j=1 \wedge [r_j,d_j] \subseteq [s,u]\}|$

The ILP solves the problem

- By Hall's theorem we can assign jobs to slots
- Simply EDD assign large jobs, then small jobs in remaining slots



- $\forall t \quad 0 \leq Y_t \leq 1$

  $\forall s,u \quad \sum_{t \in [s,u)} Y_t \geq A_{su}$

  $\forall s,u \quad u - s - \sum_{t \in [s,u)} Y_t \geq \lceil B_{su}/3 \rceil$

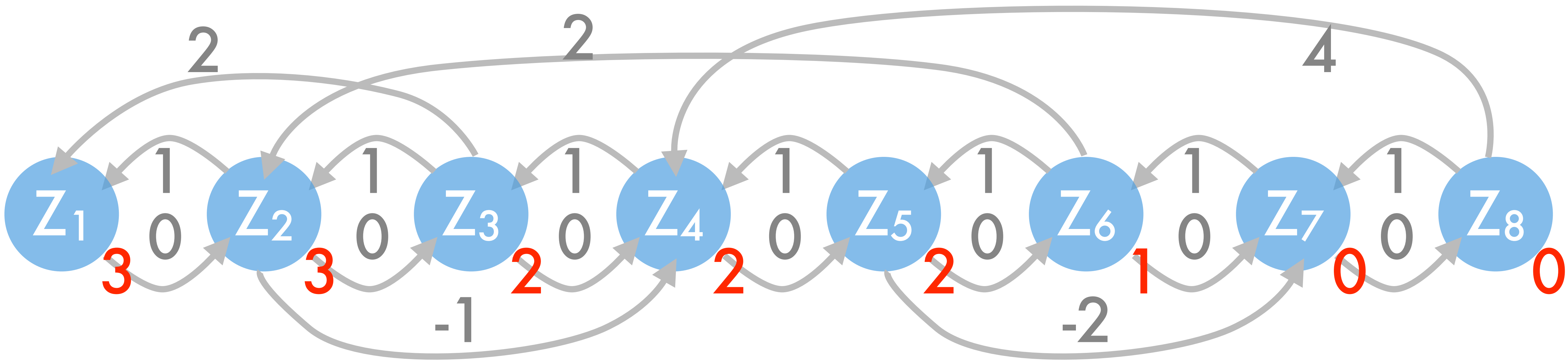
- where

  $A_{su} := |\{j: s_j = 3 \wedge [r_j, d_j] \subseteq [s,u]\}|$

  $B_{su} := |\{j: s_j = 1 \wedge [r_j, d_j] \subseteq [s,u]\}|$

The LP is totally unimodular

- additional transformation: $Z_t = Y_t + \ldots + Y_{n-1}$, $Z_n = 0$
- the LP is a shortest path problem



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  $\forall s,u$ $\quad u - s - \sum t \in [s,u)\ Y_t \geq \lceil B_{su}/3 \rceil$

- $\forall t$ $\quad 0 \leq Z_t - Z_{t+1} \leq 1$

  $\forall s,u$ $\quad Z_s - Z_u \geq A_{su}$

  $\forall s,u$ $\quad Z_s - Z_u \geq \lceil B_{su}/3 \rceil$

Where our method worked

- $3 \mid r_i, size_i \in \{1,3\}, p_i=1 \mid L_{max}$

  [Baptiste,Schieber'05] LP(n^2 var.)
  [D,Hurand'06] comb. alg. in $O(n^3)$.

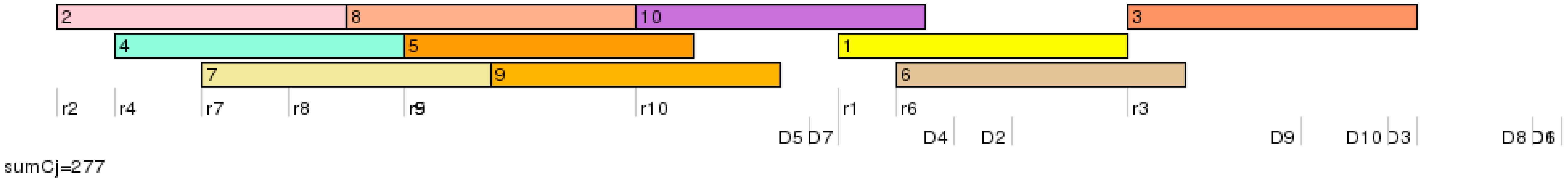
- *Offline caching with prefetching*
  [Albers,Garg,Leonardi'00] LP(n^2F var.)
  [D,Hurand'06] comb. alg. in $O(n^3)$.

requests	a	b	c	d	a	b	e	d	b	a
evicted entering pages				c>d		a>e		e>a		
stall time				4		3		2		
cache	a b c	a b c	a b c	a b d	a b d	b d	e b d	b d	b d	a b d

fetch intervals

Where our method failed

- $P \mid r_i, p_i = p, D_i \mid \sum w_i C_i$
  [Brucker,Kravchenko'05] LP(n^3 var.)



LP which we suspect to have always an integral optima

- $1 \mid r_i, p_i = p, pmtn \mid \sum w_i C_i$
  ☹ Baptiste,Chrobak,Hurand,Sgall,...
- we know that there are always integral optima when $p > \max r_i$
- we don’t know even for $r_i = i, p = 2$