The expanding search ratio of a graph

Spyros Angelopoulos	CNRS, Sorbonne Universités, UPMC
Christoph Dürr	CNRS, Sorbonne Universités, UPMC
Thomas Lidbetter	London School of Economics → Rutgers

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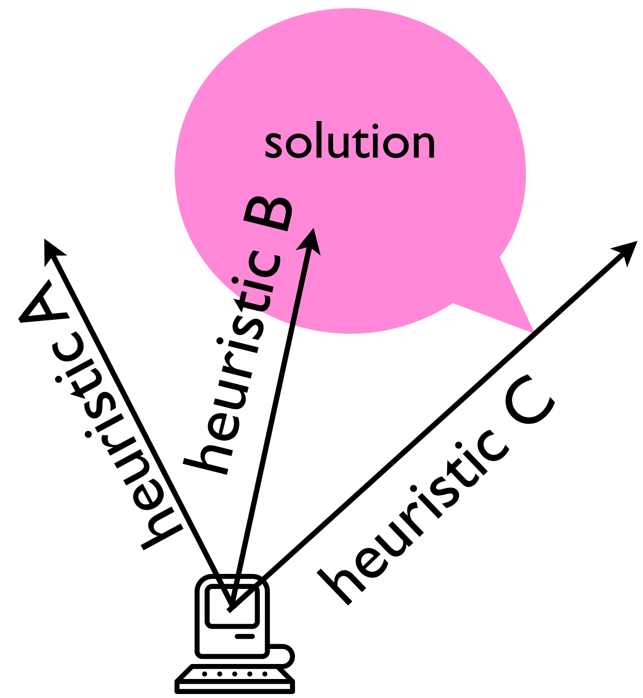
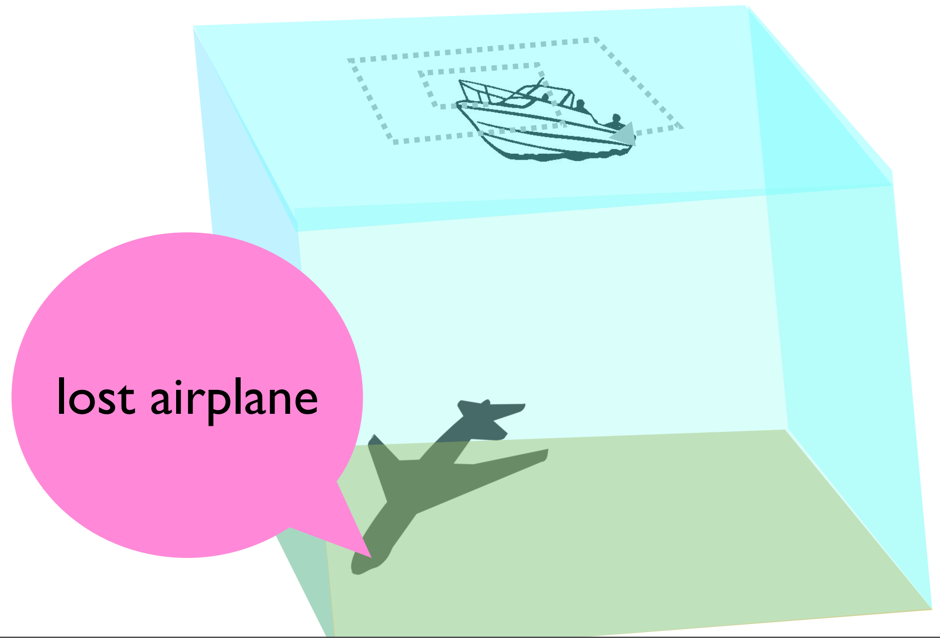
Outline

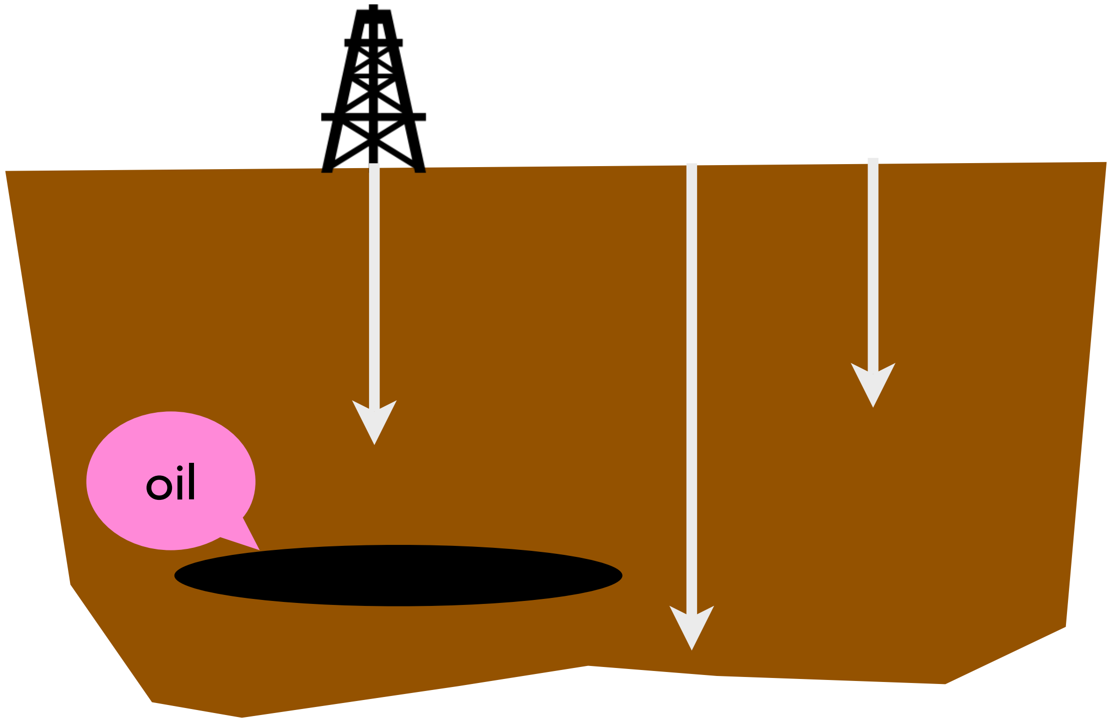
- Introduction
- Warmup: the cow-path problem
- The deterministic expanding search ratio on general graphs
- The randomized expanding search ratio on trees
- The randomized expanding search ratio on star graphs

Search problems

- **You know:** the search space
- **You don't know:** where the hidden item is
- **Decide:** how to explore the search space as efficiently as possible
- **Many** Applications

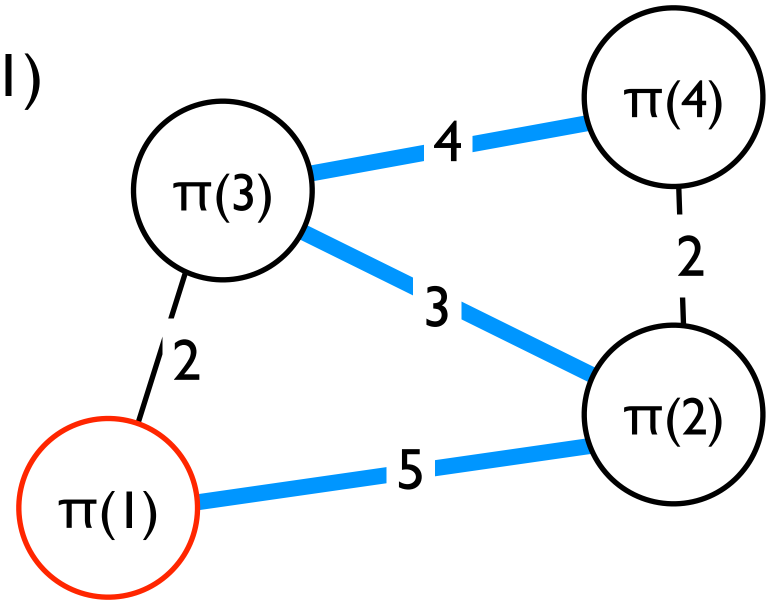






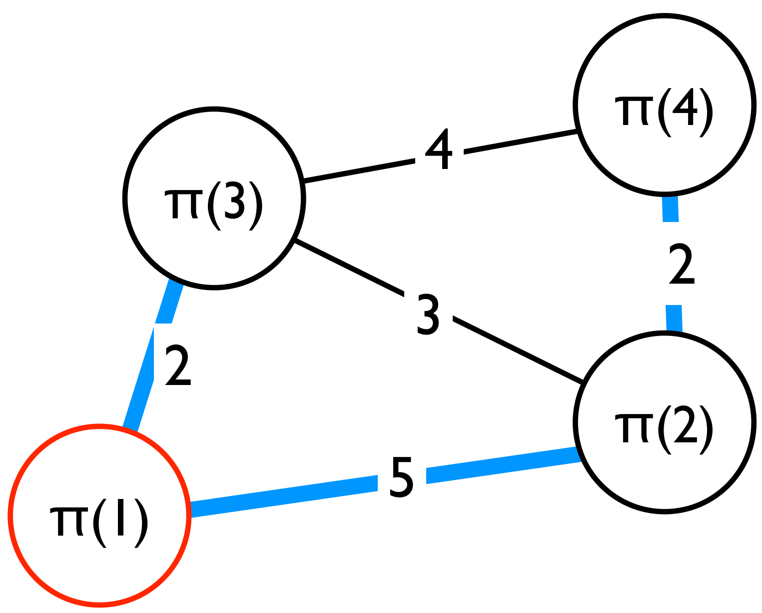
2 search models

- **A search strategy:** is an ordering π on vertices with π(1) = origin O
- **Pathwise search:** connect π(i) to π(i-1)



path

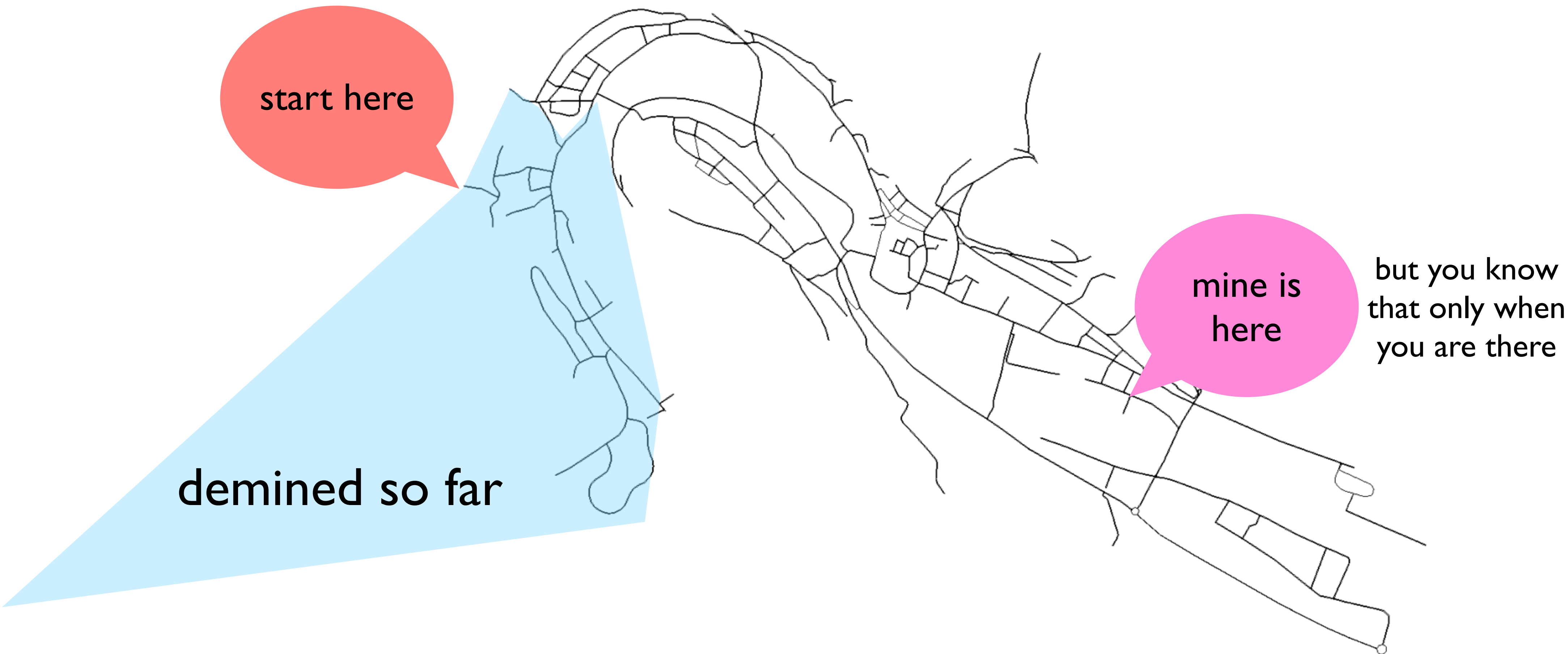
- **Expanding search:** connect π(i) to the closest among π(1),..., π(i-1)



tree

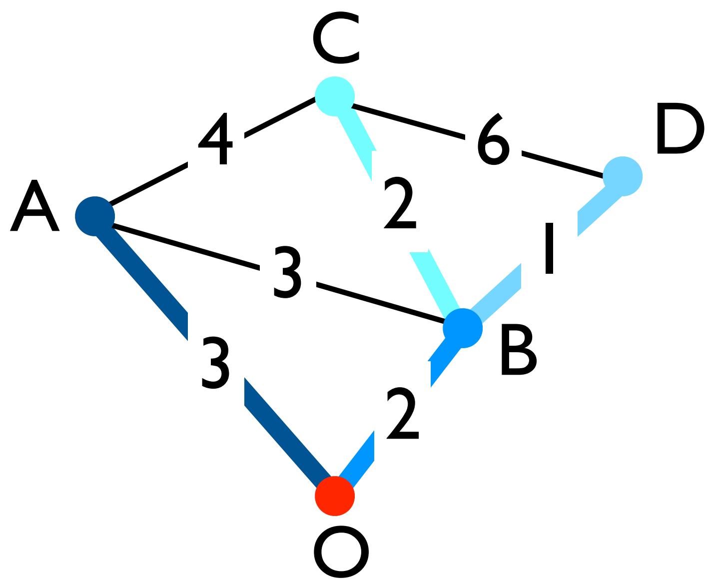
a motivation for expanding search

- **You know:** A single mine is set in some crossing of a road network
- **You don't know:** where
- Demining a road segment takes some time
- Walking along demined roads takes no time
- In what order should you proceed ?



2 cost models

- **Given search strategy π:** every vertex v is visited at some time T(v) (= *search time)*
- **Average search time:** $$\min_{\pi} \frac{1}{|V \setminus O|} \sum_{v \in V \setminus O} T(v)$$
- **Search ratio:** $$\min_{\pi} \max_{v \in V \setminus O} \frac{T(v)}{d(O,v)}$$



search time		distance	search ratio
O	0	0	
A	3	3	1
B	3+2=5	2	2.5
D	5+1=6	3	2
C	6+2=8	4	2
	5,5		

- **randomized variant:** distribution over search strategies that minimizes expected cost

Some previous work on graphs

	Average search time	Search ratio
Pathwise search	also called *minimum latency* or *traveling repairman problem* Blum Chalasani Coppersmith Pulleyblank Raghavan Sudan 1994	Koutsoupias Paradimitriou Yannakakis 1996
Expanding search	Alpern Lidbetter 2013	This work

Results

	edge weighted graph	unweighted graph or edge weighted tree	edge weighted star graph
deterministic search ratio	NP-hard 5.55 approximation	polynomial	
randomized search ratio		1.25 approximation	polynomial [Condon et al'2009]

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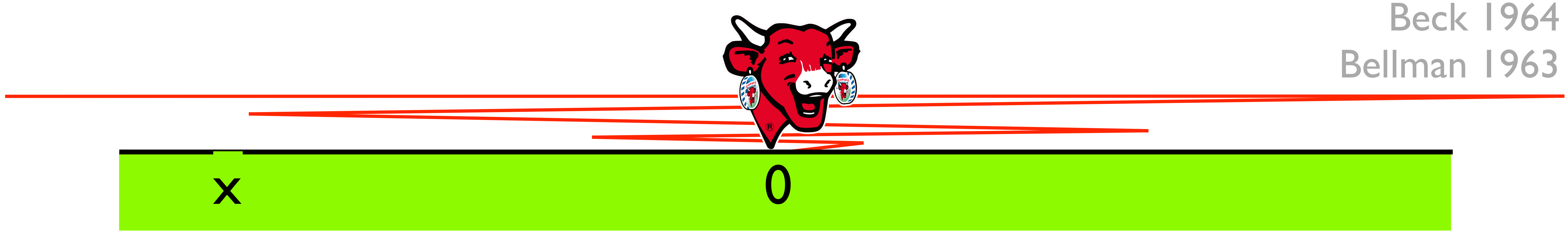
cow path problem: definition

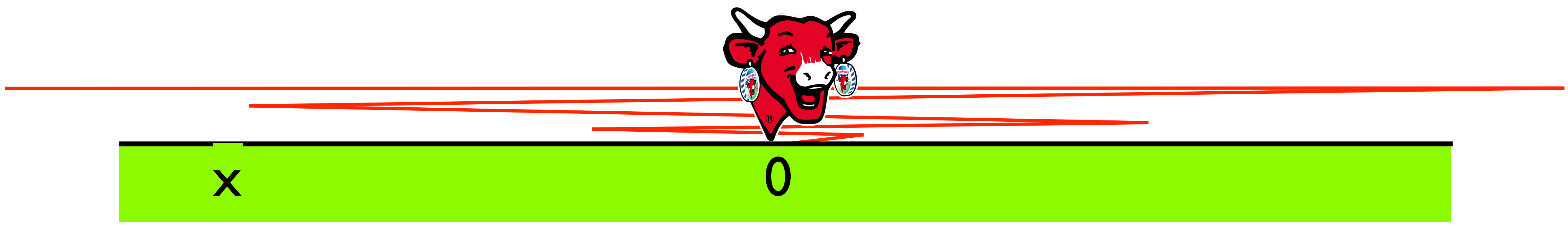
Beck 1964
Bellman 1963



- there is juicy grass on the other side of the fence
- cow is at position 0,
  fence has an opening at position x with $|x| \geq 1$
  (sign(x) is unknown to the cow)
- doubling ALG: walk to +1,-2,+4,-8,+16,-32,.
- competitive ratio:= distance walked to ope

randomized 4.5911
Kao, Reif, Tate 1996

cow path problem: analysis



- worst case: $|x|=2^i+\varepsilon$
- cost of ALG:

$$2(1+2+4+\ldots+2^{i+1}) + 2^i+\varepsilon$$
$$= 2(2^{i+2}-1) + 2^i+\varepsilon$$
$$< 8\cdot 2^i + 2^i+\varepsilon$$
$$< 9\cdot \mathrm{OPT}$$

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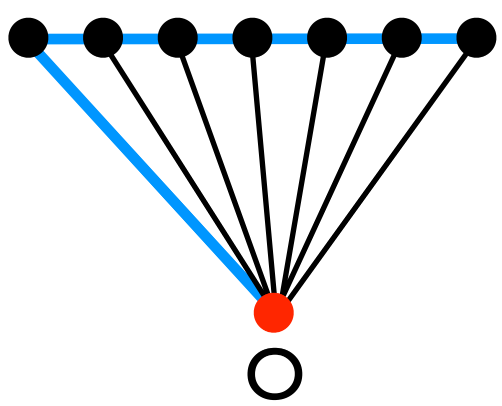
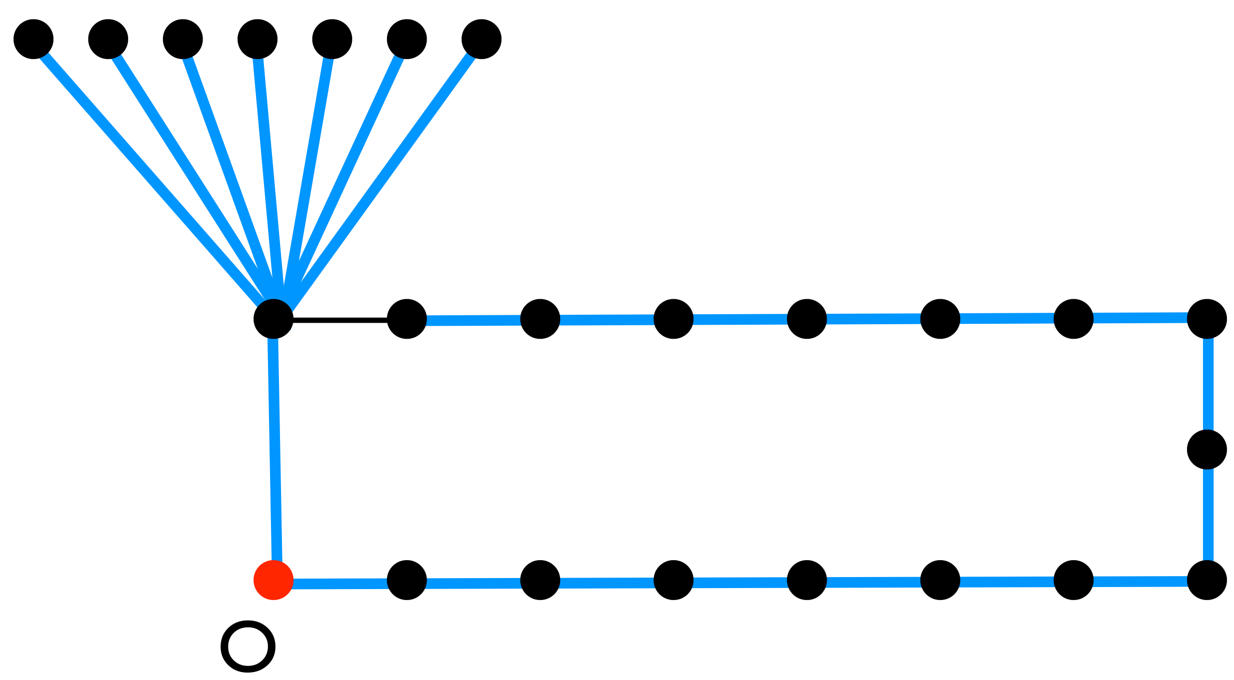
Get some intuition

- Suppose we give ourself an acceptable upper bound on the search ratio we want to attain
- Then we have a deadline (=UB on ratio times distance) for visiting each vertex

don't expand simply along
a shortest path tree

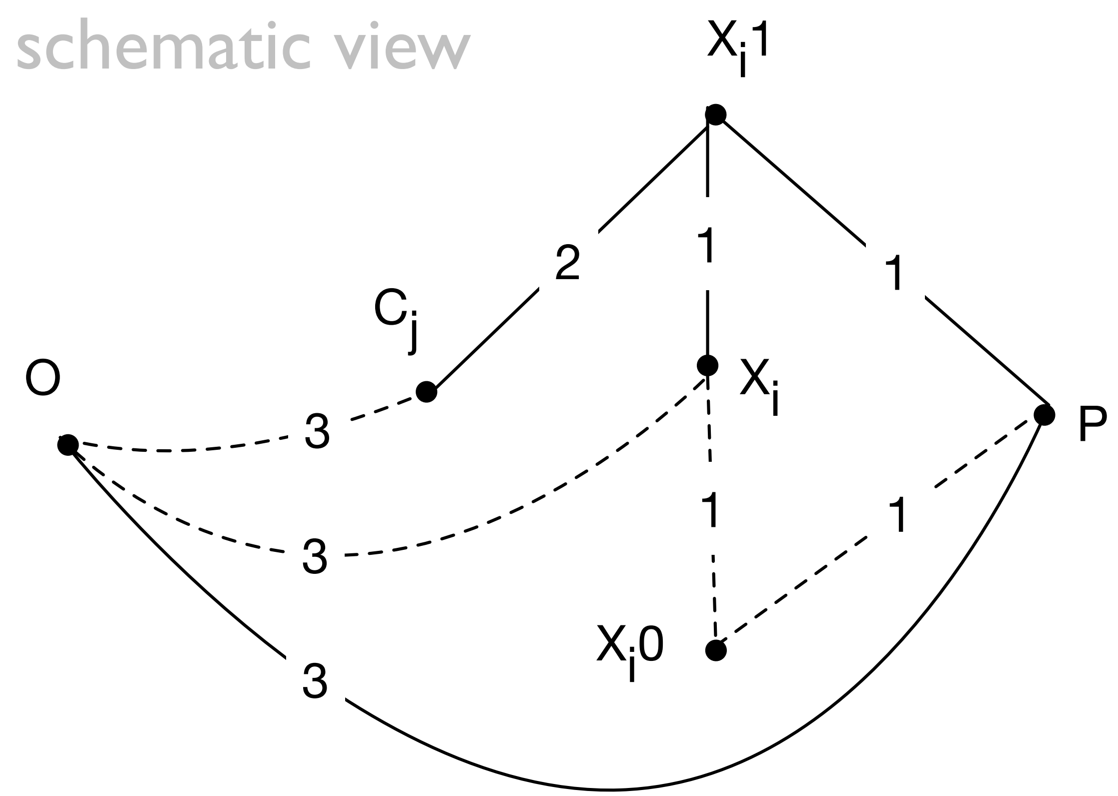


don't expand simply along
a minimum spanning tree



NP-hardness

schematic view



- Computing the optimal search ratio is NP-hard
- Reduction from 3-SAT
- There are clause vertices, variable vertices, literal vertices, origin O, single vertex P
- If there is a satisfying assignment then there is an expanding search of ratio

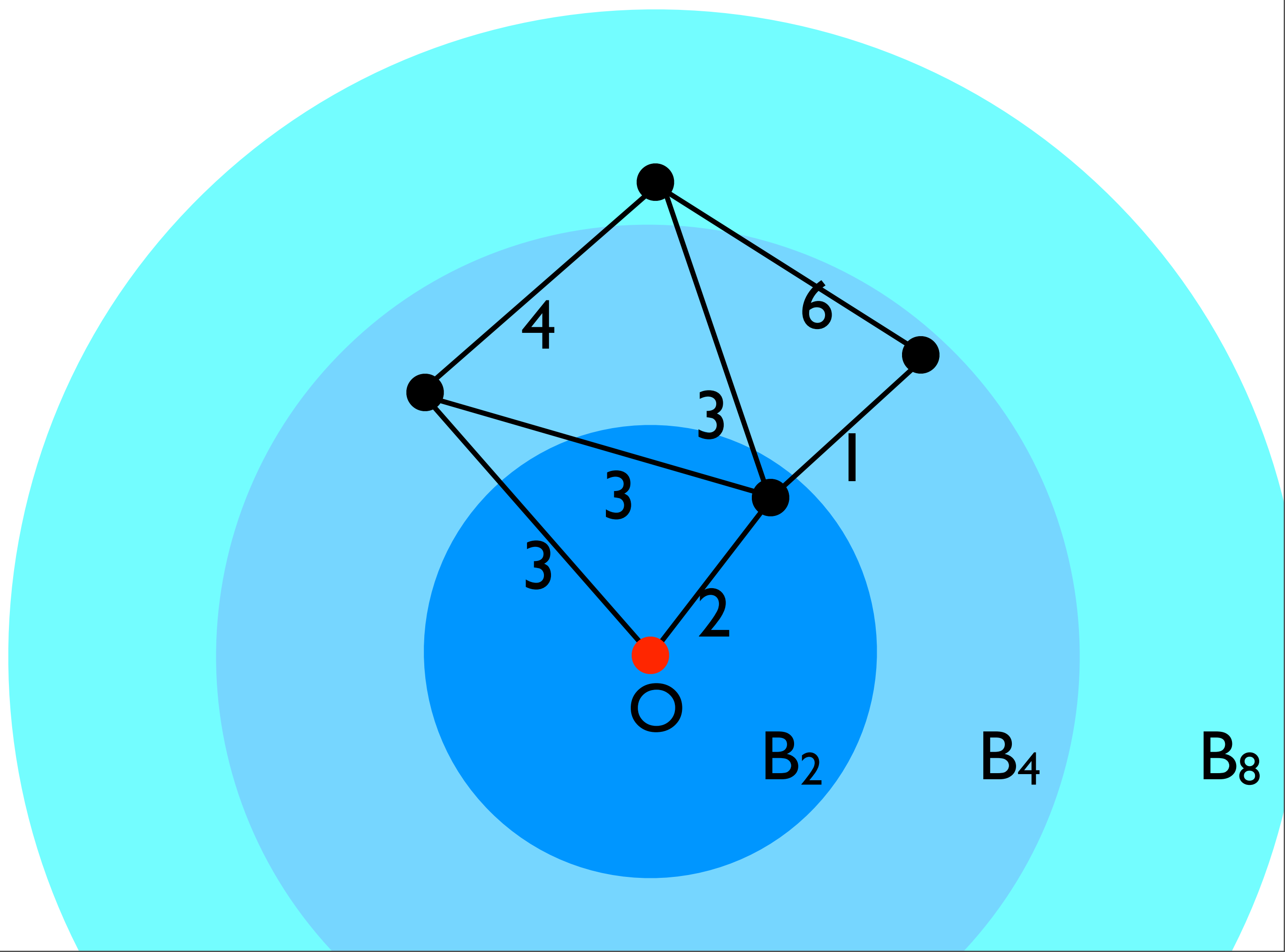
  $1+\frac{2}{3}(\#\text{variables}+\#\text{clauses})$

  and vice-versa

A 5.55 approximation

Algorithm

- sol = []
- For doubling radii d (1,2,4,…) do
  - let B_d be the ball of vertices v with distance at most d from O
  - let $\hat{G}_d$ be a ln(4) approximation of the minimum Steiner tree of B_d (Opt = G_d)
    Byrka Gradoni Rothvoß Sanità 2010
  - append to sol an abitrary expanding search of $\hat{G}_d$ (omitting edges useless for connectivity)



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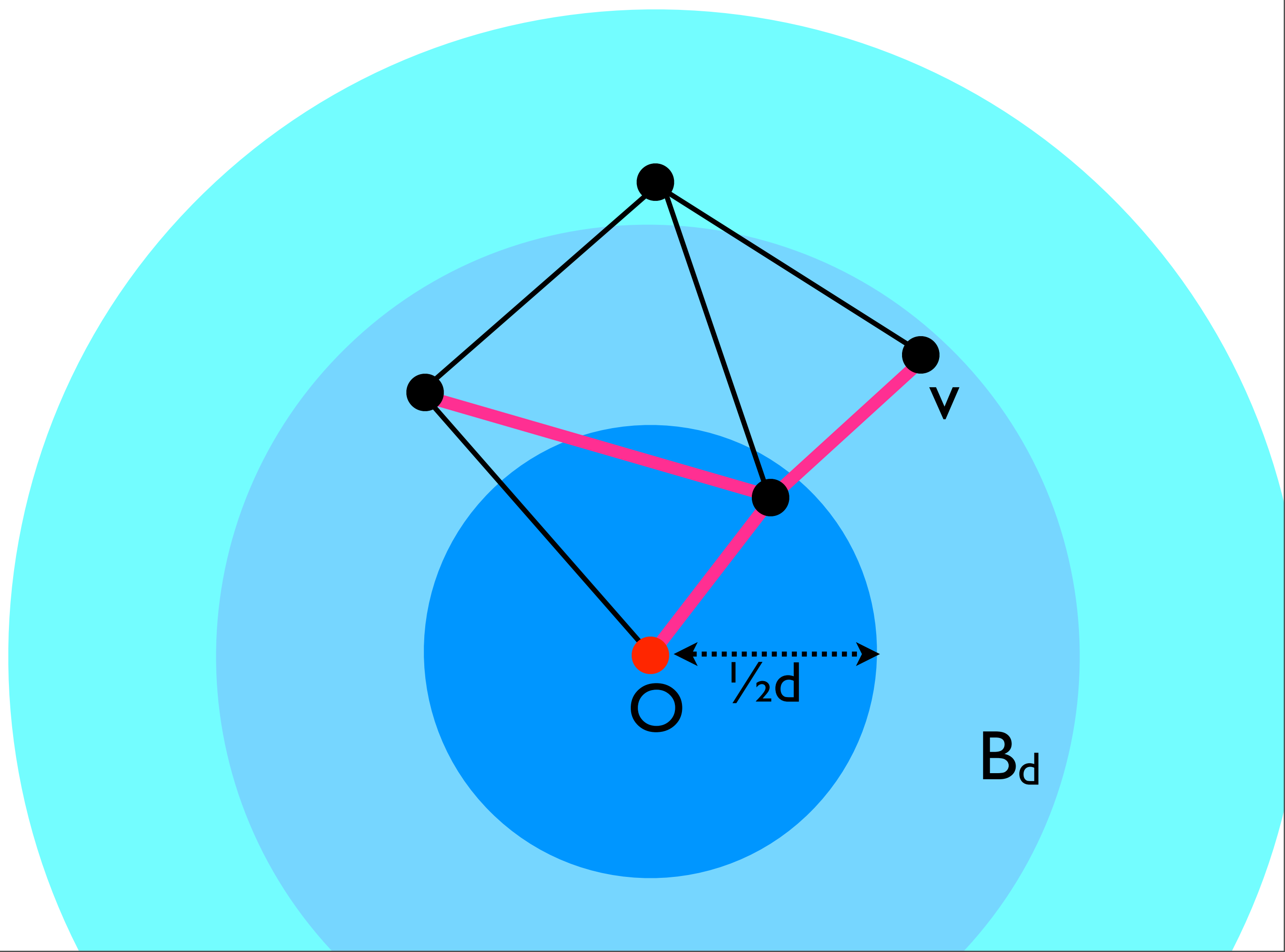
Analysis

- Consider worst target v is at level d
- Search ratio
  = w(first tree hitting v) / dist(O,v)
  $\leq \Sigma_{k=1,2,4\ldots d}\, w(\hat{G}_k) / \frac{1}{2}d$
  $\leq \ln(4)\, \Sigma_{k=1,2,4\ldots d}\, w(G_k) / \frac{1}{2}d$
  $\leq \ln(4)\, \text{OptRatio}\, \Sigma_{k=1,2,4\ldots d}\, k / \frac{1}{2}d$
  $\leq 4 \ln(4)\, \text{OptRatio}$



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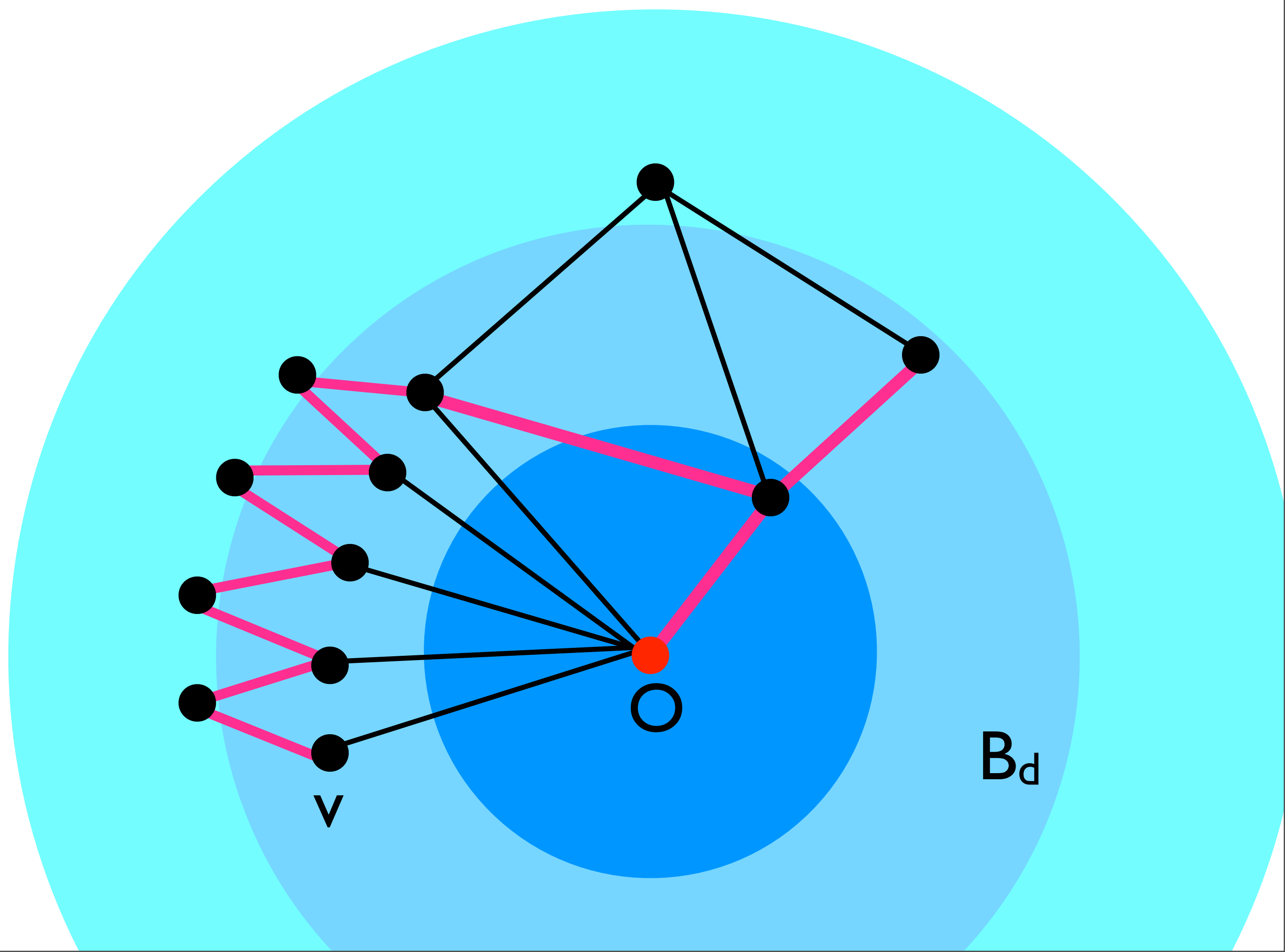
Key argument

- Minimum Steiner tree G_d for B_d
- Consider Optimal expanding search
- and its first tree T that covers B_d
- then $w(T) \leq \text{OptRatio} \cdot d$
- and $w(T) \geq w(G_d)$



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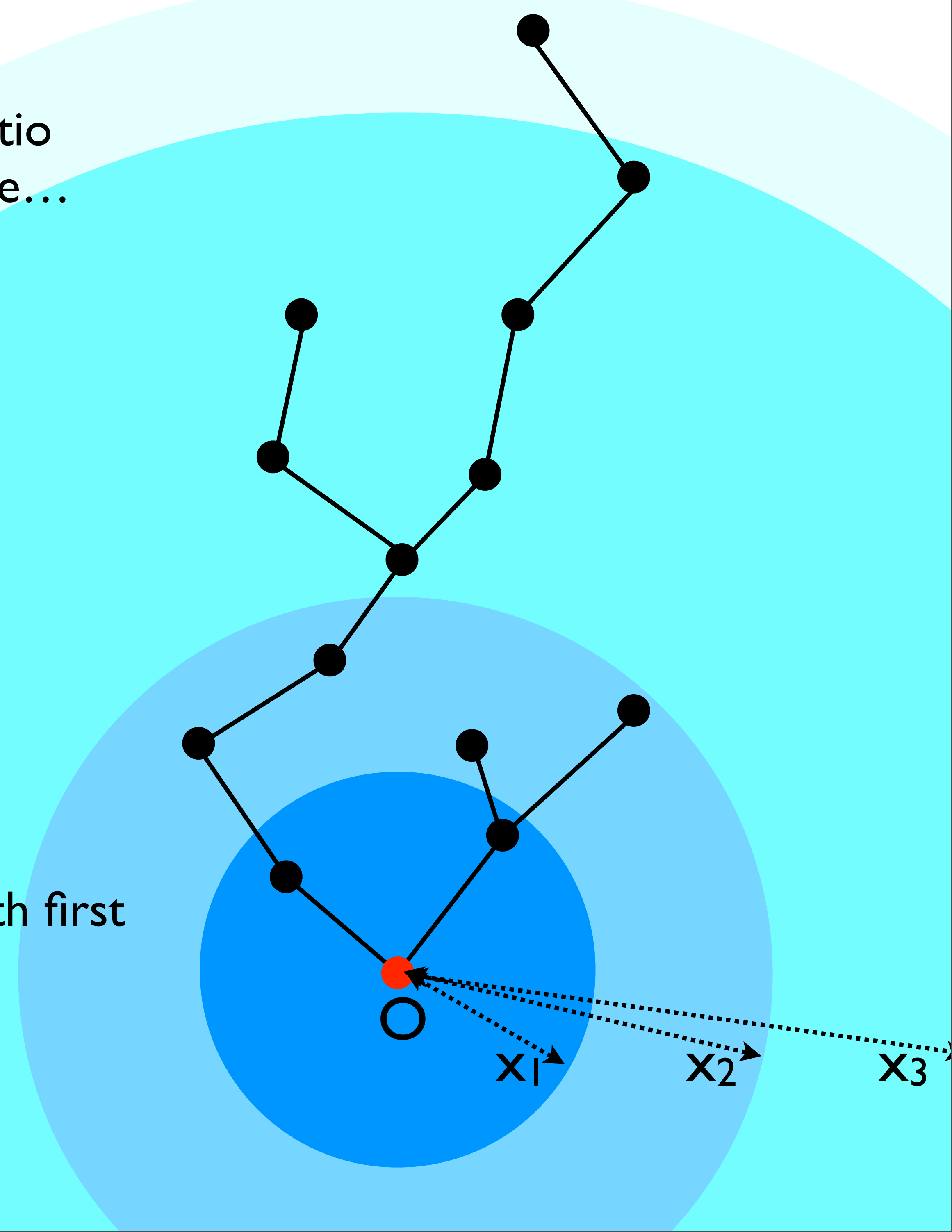
A 5/4 approximation of the randomized search ratio

- **Theorem:** ALG ≤ 5/4 OPT + 1
  for OPT = optimal expected normalized search ratio
  ALG = expected normalized search ratio of the…

- **Randomized deepening algorithm**
  - sol=[ ]
  - T={O}
  - For i=0,1,2… do:
    - choose x_i unformly at random in $[2^{i-1},2^i]$
    - add to sol all vertices with dist[T,v]≤x_i in depth first search order with random child order
    - T=sol



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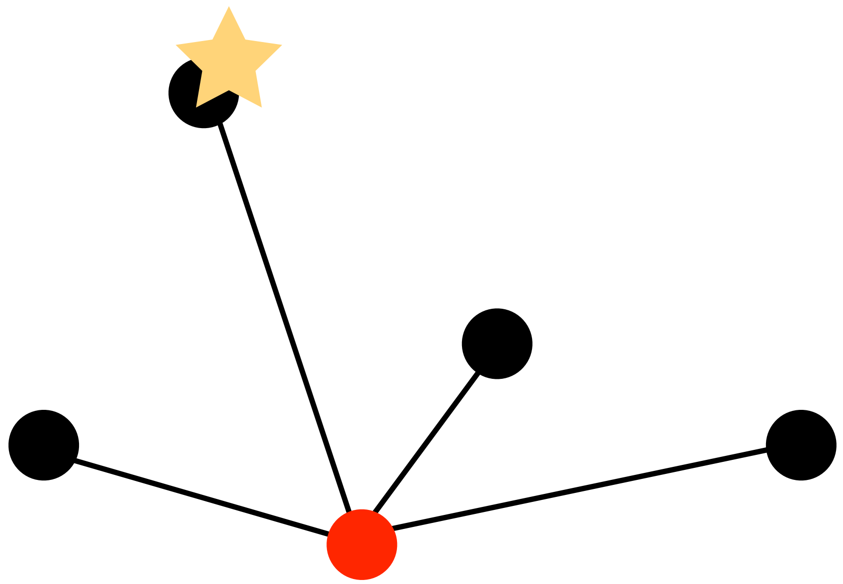
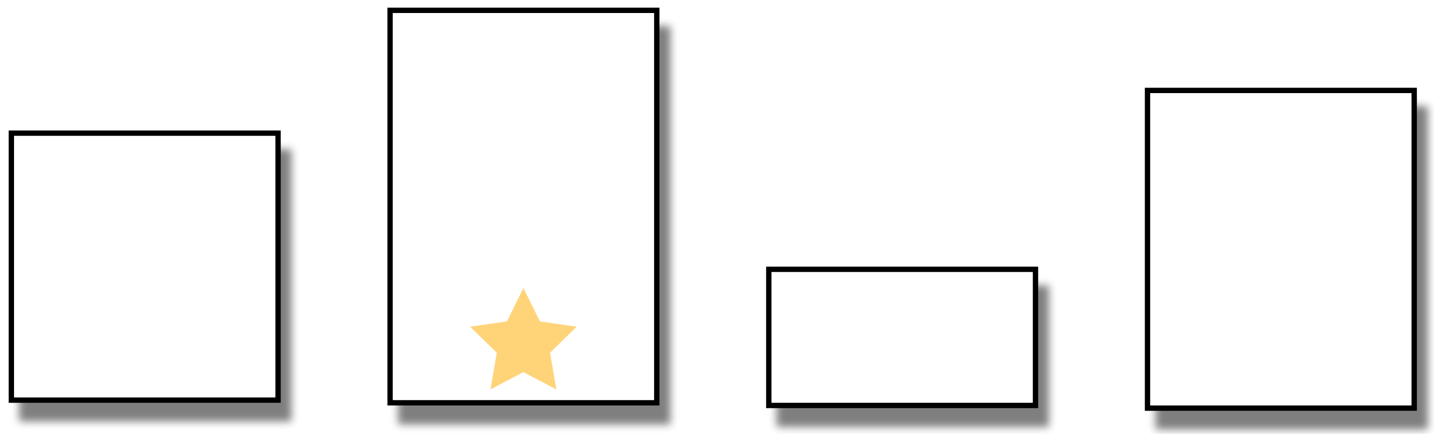
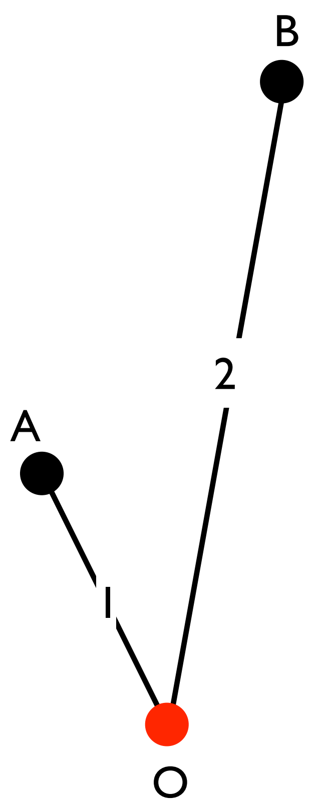
Expanding search on star graphs



Opening times of the boxes are known.
What probability distribution on box orders should we follow to minimize the time to find the star ?



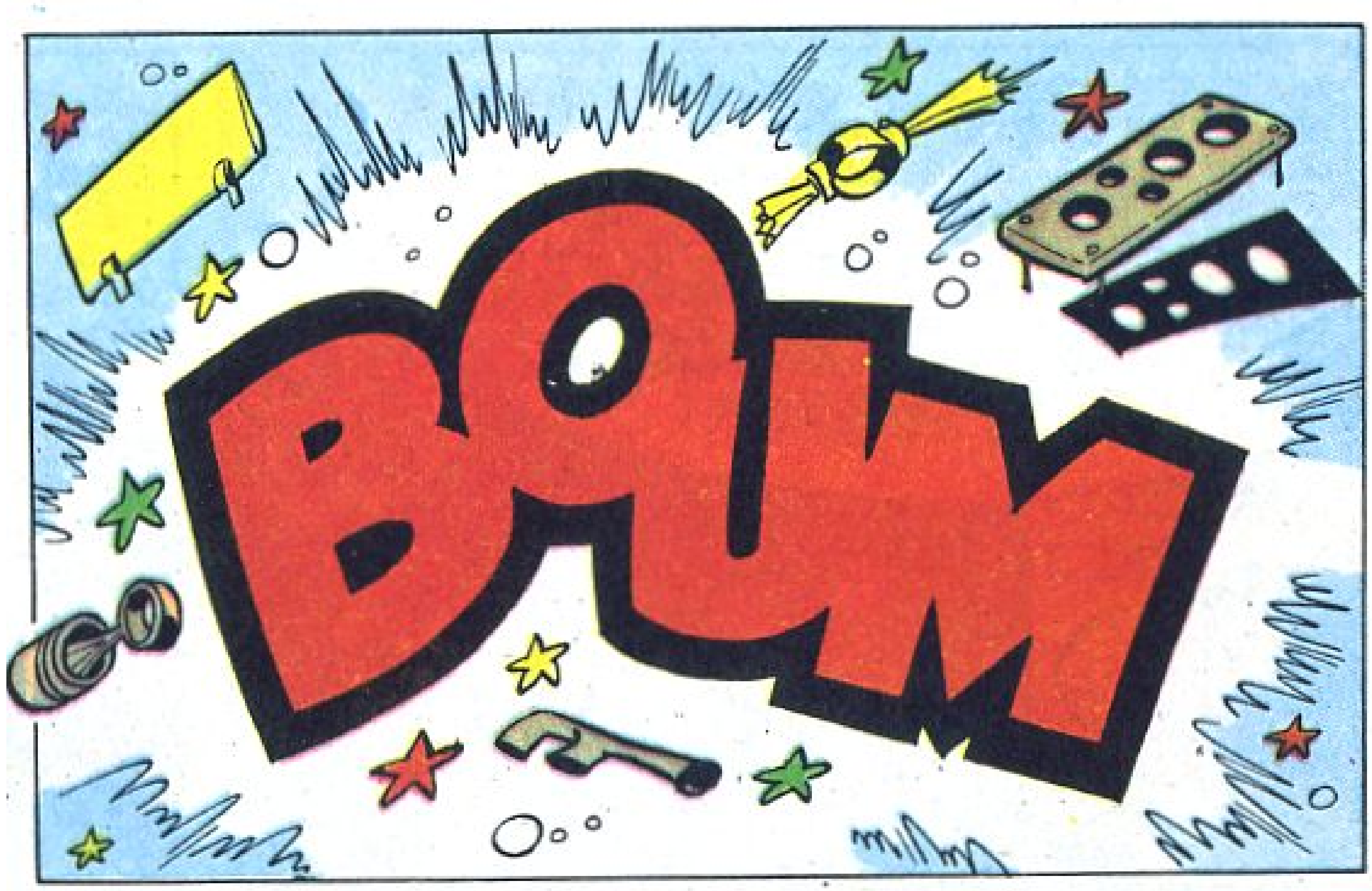
Example



probability	4/5	1/5	expected search time
Search order	AB	BA	
target at A	1	3	$\frac{4}{5} \cdot 1 + \frac{1}{5} \cdot 3 = \frac{7}{5}$
target at B	3/2	2/2	$\frac{4}{5} \cdot \frac{3}{2} + \frac{1}{5} \cdot \frac{2}{2} = \frac{14}{10}$

Results

- We recursively define a strategy that is optimal in some case
- Condon et al. 2009 gave an $O(n^2)$ algorithm to compute an optimal distribution



Thank you