Competitive Strategies for Online Clique Clustering

Christoph Dürr

joint work with

Marek Chrobak

Aleksander Fabijan

Bengt J. Nilsson

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Online graph arrival model

Vertices arrive one by one, revealing edges to existing vertices.

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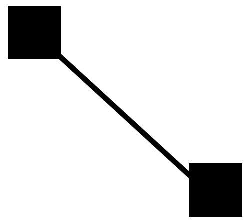

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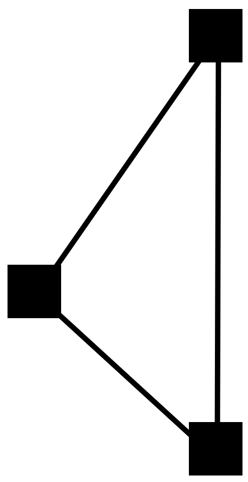

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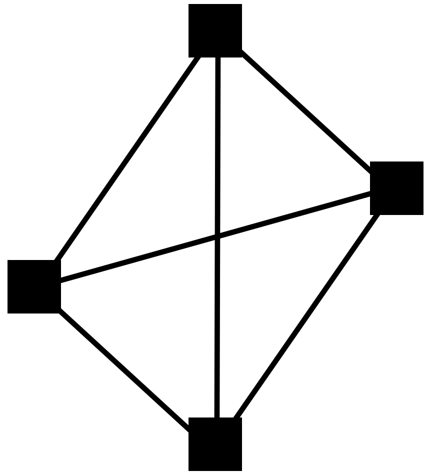

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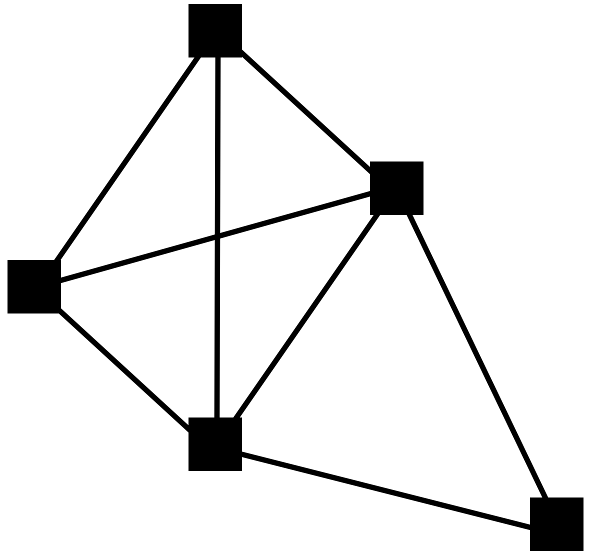

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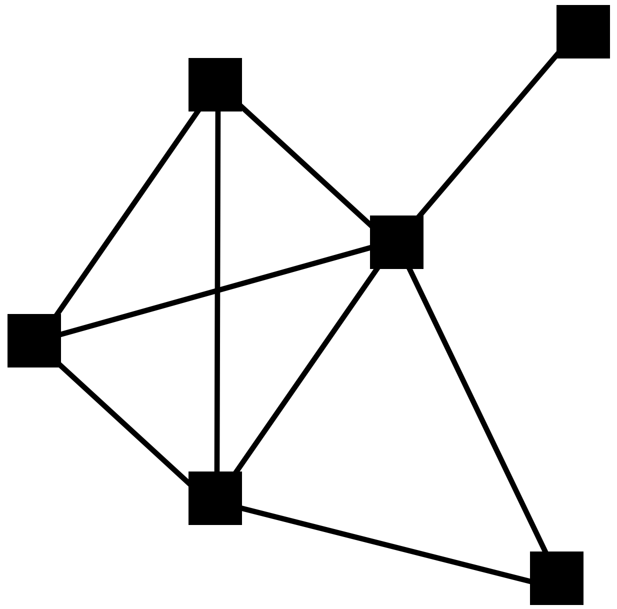

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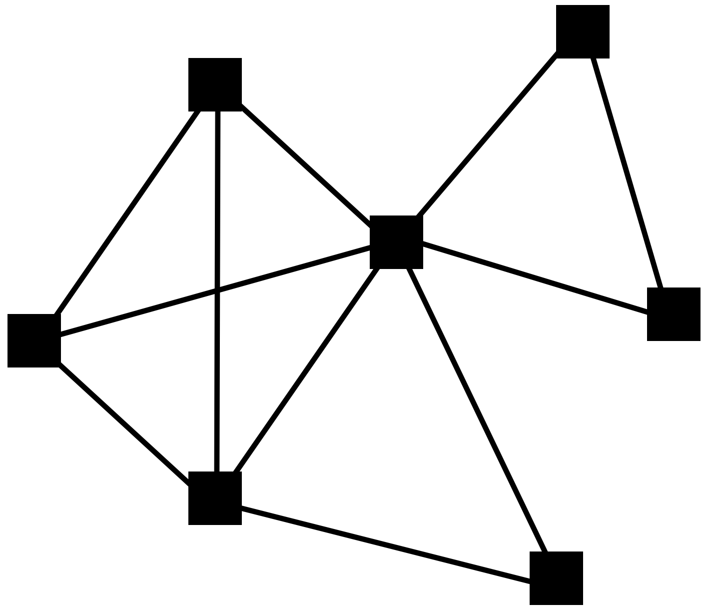

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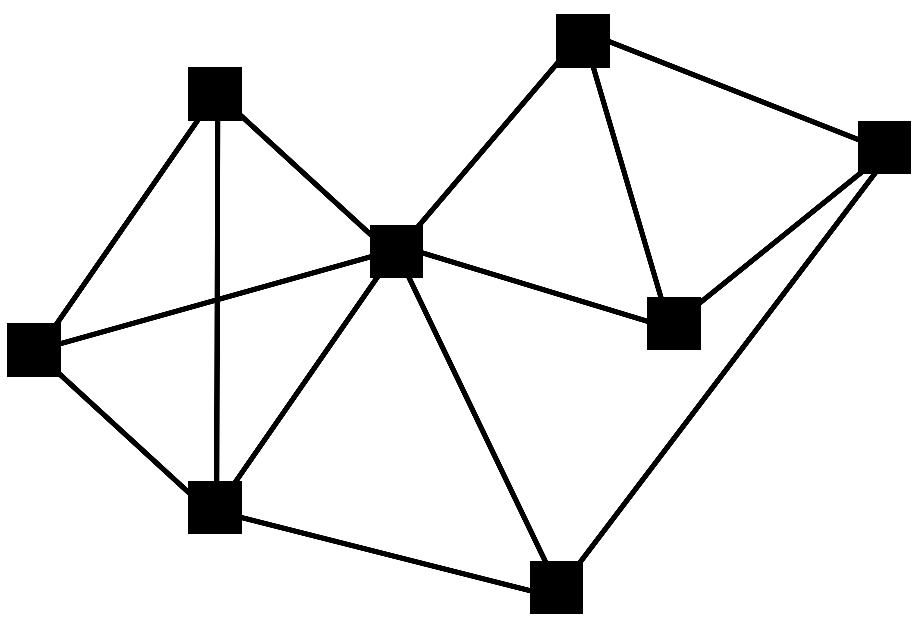

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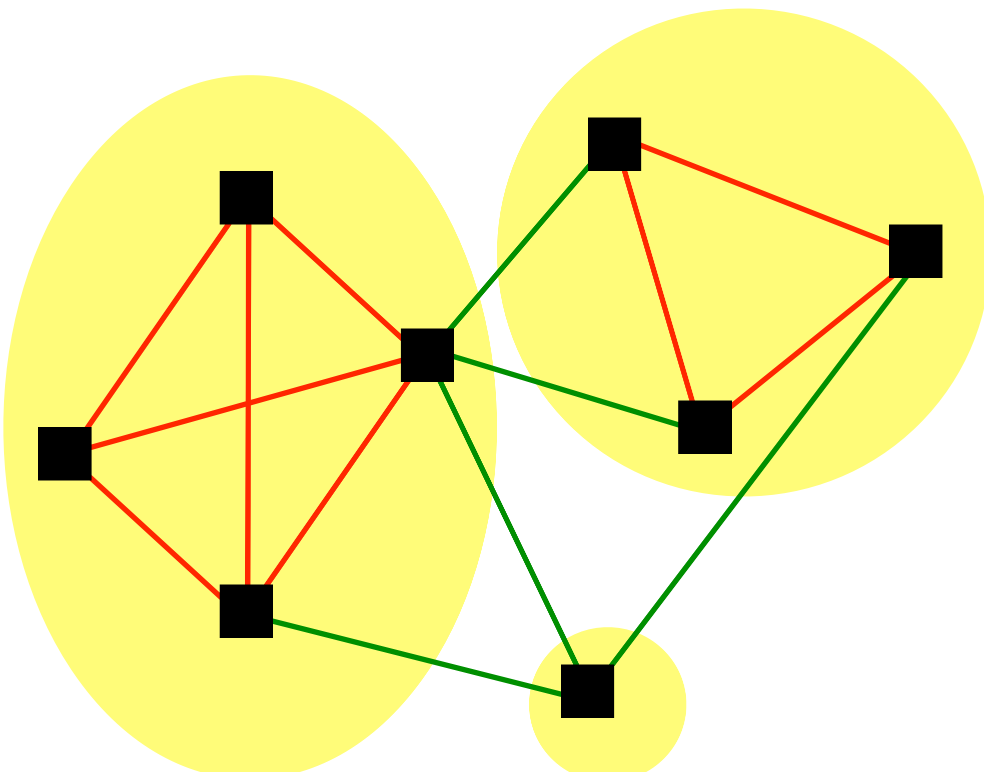
Irrevocable decisions have to be made at each step.

Partition a graph into cliques



contribution of a clique of size k is $\binom{k}{2}$

Decide:
maintain partition into cliques by merging. Splitting not allowed.

Objective values:

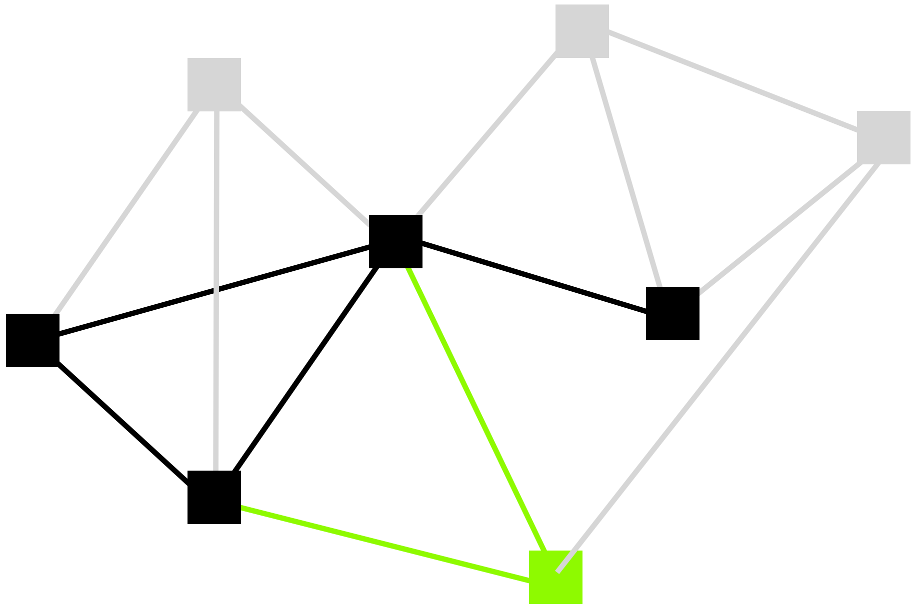
MaxCC
maximize number of edges inside the cliques

MinCC
minimize number of edges between the cliques

Motivation: DNA clone classification, gene expression profiling

Offline problem: not approximable within a factor of $n^{1-o(\varepsilon)}$ under some complexity hypothesis

Online model



- vertices arrive one by one, revealing adjacent edges
- strategy is allowed to merge cliques, but can never split them again
- **competitive ratio**: between obj. value of algorithms solution over optimal obj. value for current graph.

competitive ratio	LB	LB for doubling technique	UB
MaxCC	6	10.927	22.641
MinCC	n-2		n-2

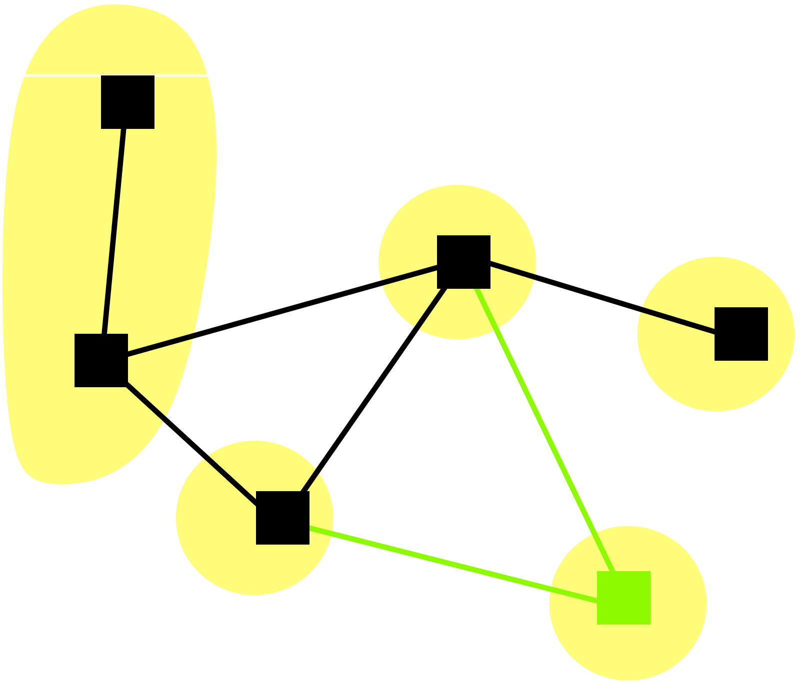
- There was a first paper by Fabijan, Nilsson, Persson, CIAC 2013

MaxCC

Online strategy model

- Our strategy needs an oracle that tells him the optimal clustering for some subgraph
- In that sense it is not really an online algorithm serving requests in polynomial time, but rather a "strategy"
- Hence it is ok, for the problem to have constant competitive ratio, even though the offline problem cannot be approximated in polynomial time within a constant ratio

Our strategy OCC



- Parameter $\gamma>1$
- stages, starting with $j=0$
- at each new vertex :
  - place it in a singleton clique
  - if singletons can form a clique partition with profit $\geq \gamma^j$, then merge according to this partition and move to stage $j+1$

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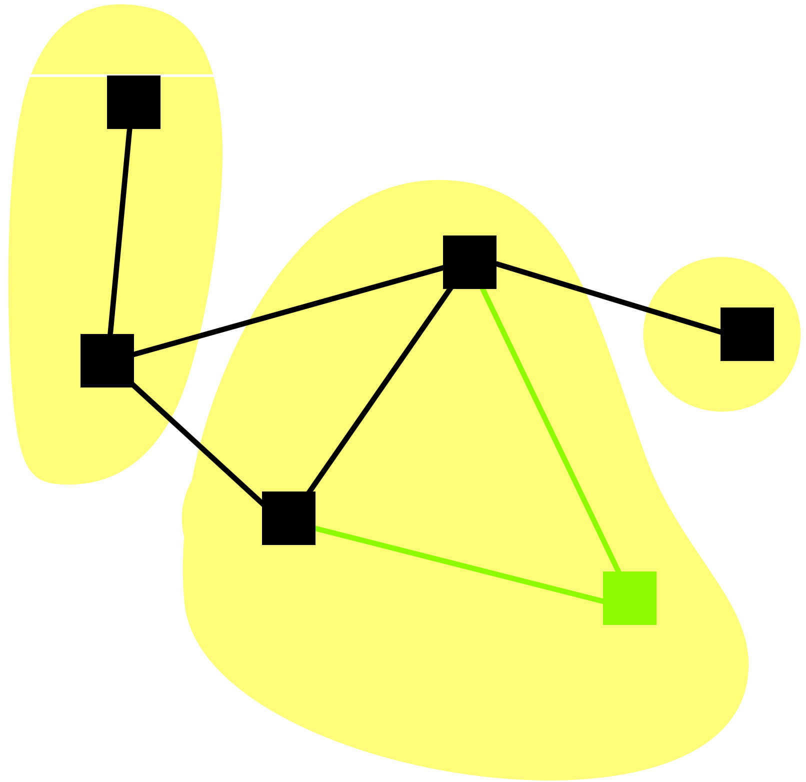

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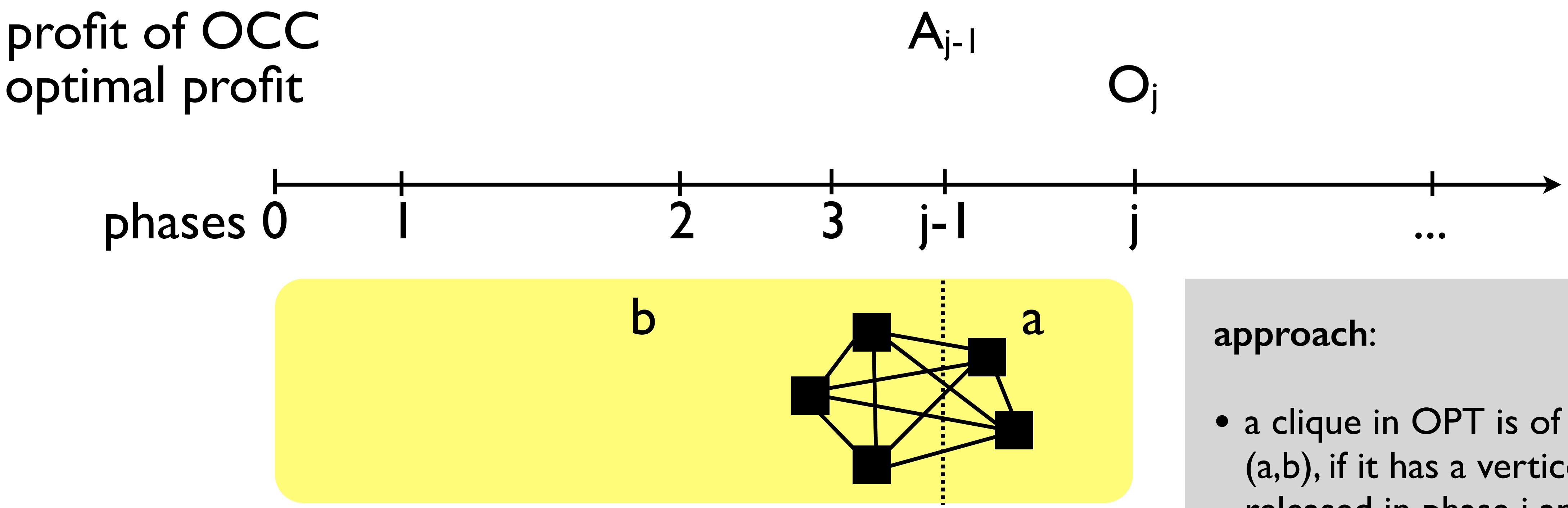
Upper bound on ratio of OCC



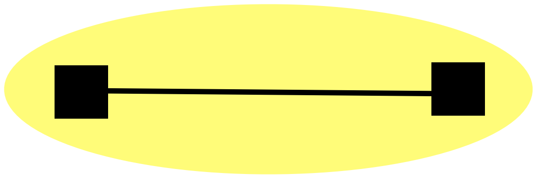
- .. is bounded by a geometric sequence
  $$A_j \geq \gamma^0 + \gamma^1 + \dots + \gamma^j \geq (\gamma^{j+1} - 1)/(\gamma - 1)$$
- ...
- we obtain a rough bound
  $$\text{assymptotic ratio} \leq (3+\sqrt{13})(5+\sqrt{13})^2 / (12(\sqrt{13}-1)) \approx 15.645$$
  for
  $$\gamma = (3+\sqrt{13})/2 \approx 3.303$$

approach:

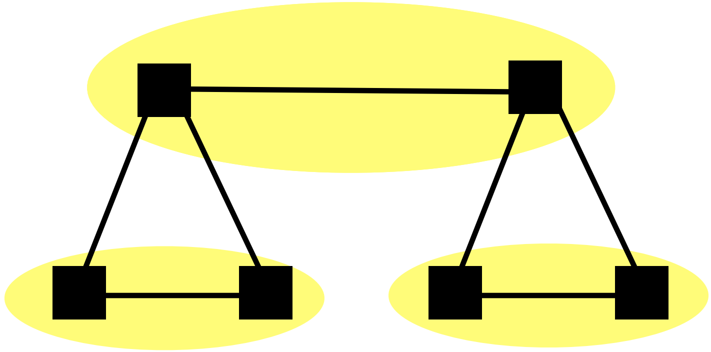
- a clique in OPT is of type (a,b), if it has a vertices released in phase j and b from previous phases.
- k_{ab} := number of cliques in OPT of this type.
- Use these numbers to bound A_{j-1}, and O_j

The hard life of an online strategy

- **Usual Game**: adversary release some vertex, strategy makes possibly some clustering, adversary releases ... (and so on, say forever)
- If strategy wants to reach competitive ratio R, he can postpone all merging until
  value of current solution * R < optimal solution for current graph



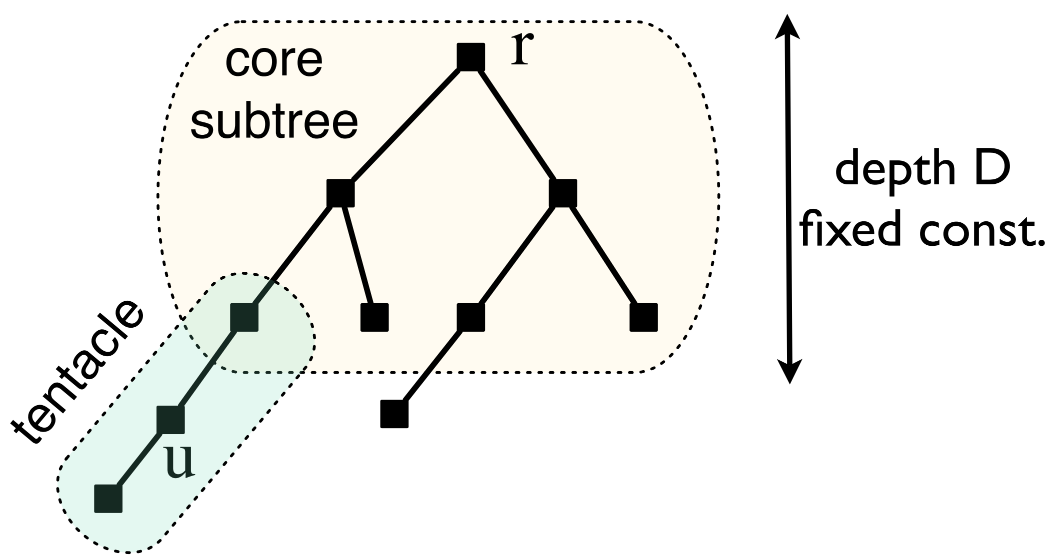
- Strategy needs to take the first edge, otherwise ratio is 1/0

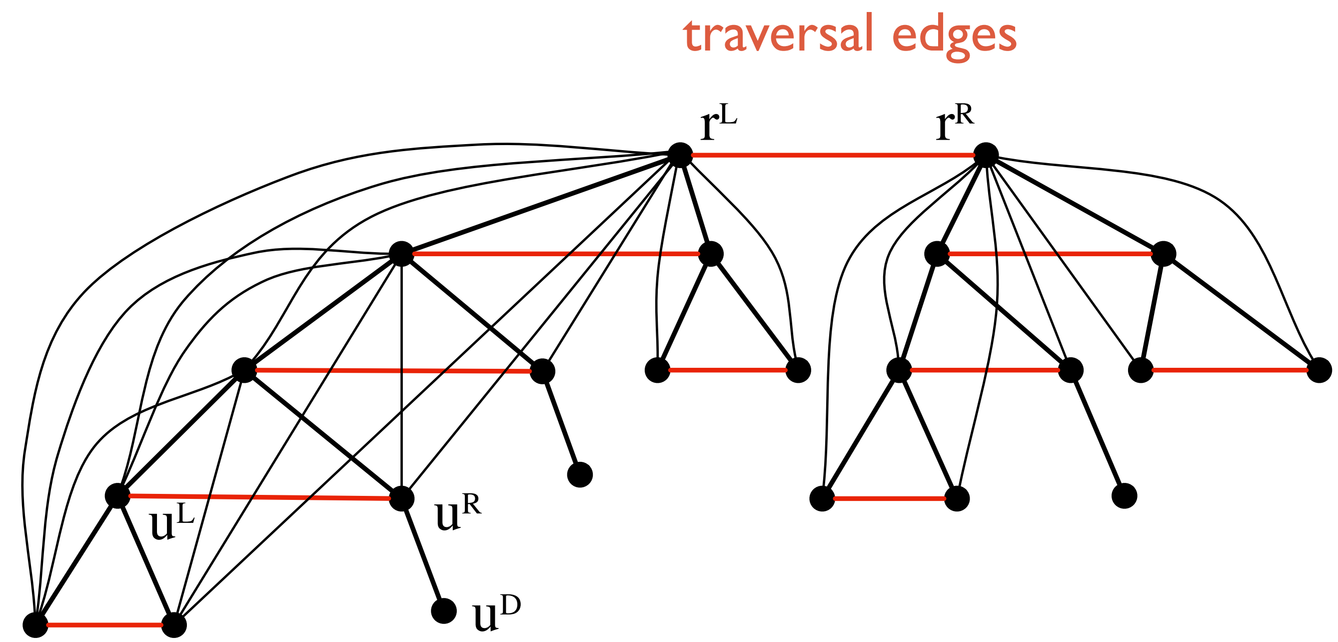


- But if endpoint is part of a clique, algo. can obtain only part of that clique. Ratio is already 6/3, let's improve it …

Lower bound of 6 on any strategy

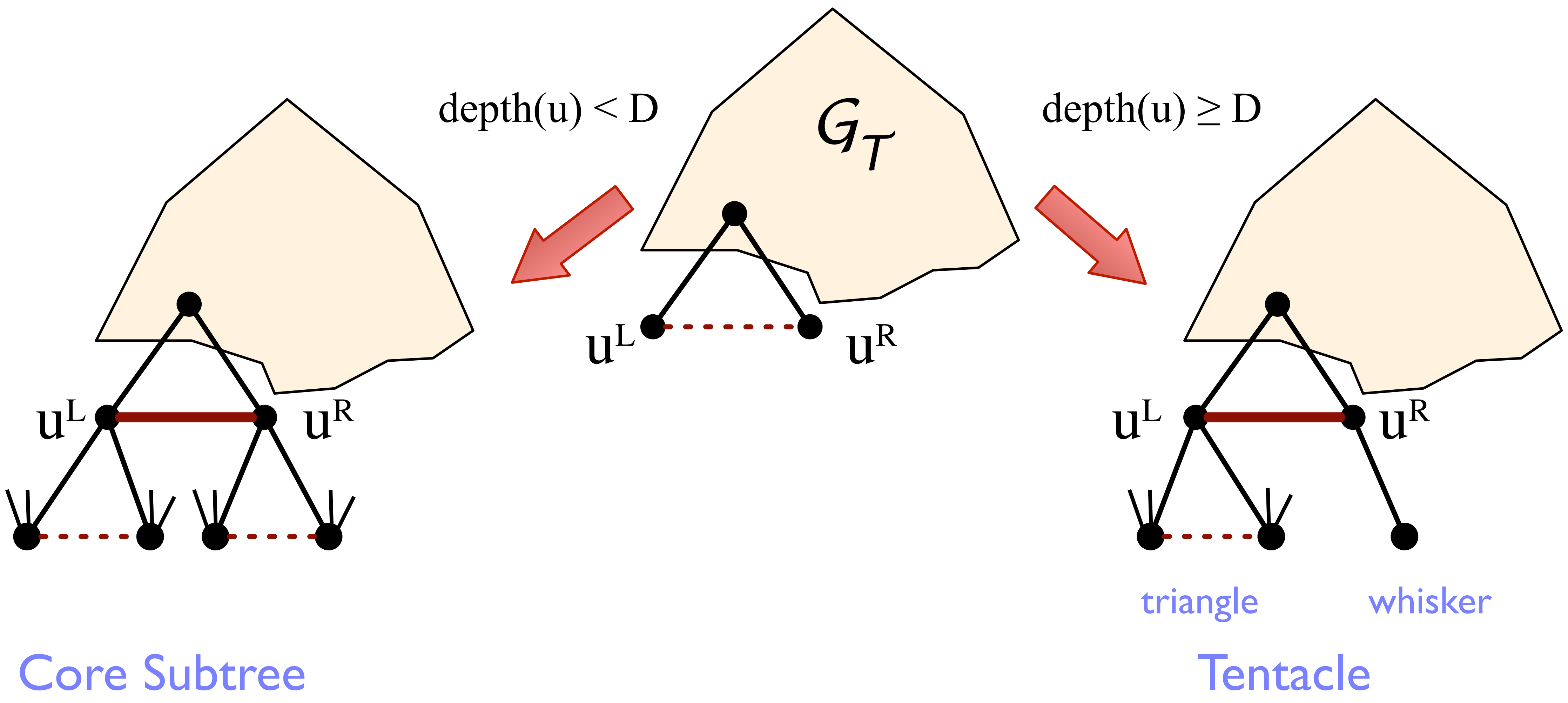
- graph generated by strategy is described by a "skeleton tree"





How the adversary extends the tree

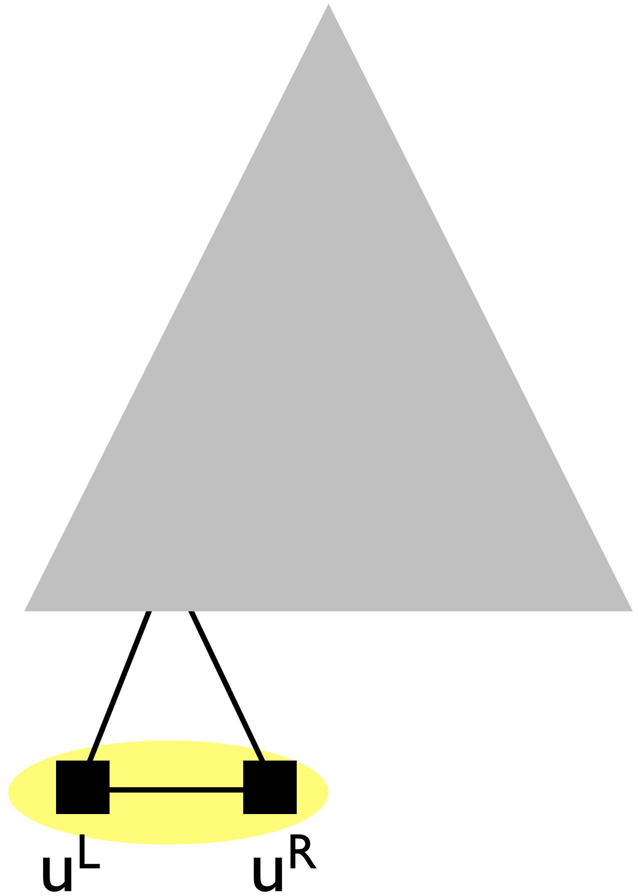
- if the strategy collects the edge (u^L,u^R)

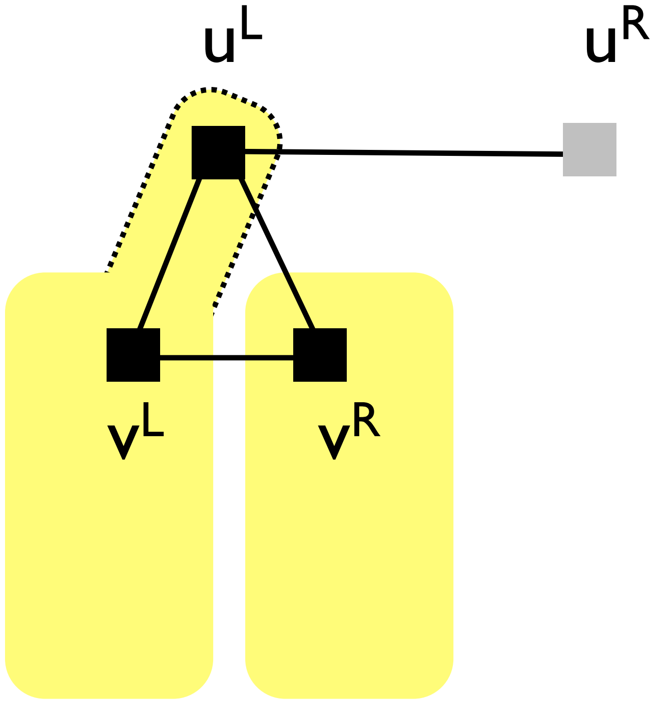


An adversarial clique partition

Bottom up description:

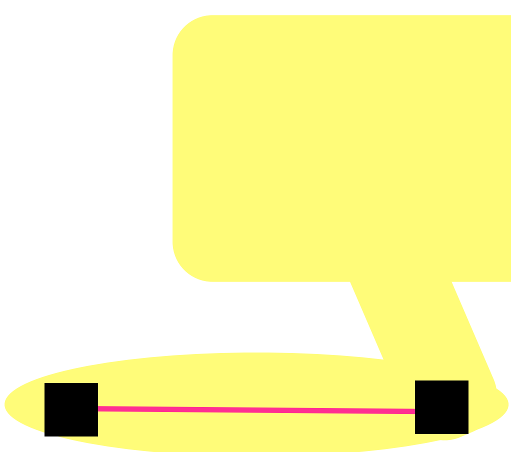
- If u is a leaf, (u^L, u^R) are in a clique
- If u has a left son v, u^L is added to one of the cliques containing v^L or V^R
- If u has a right son v, (idem)

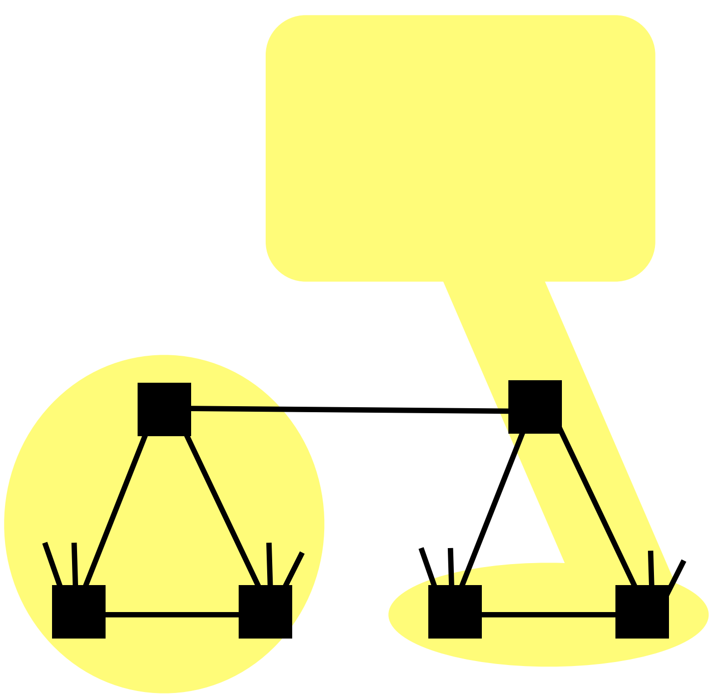


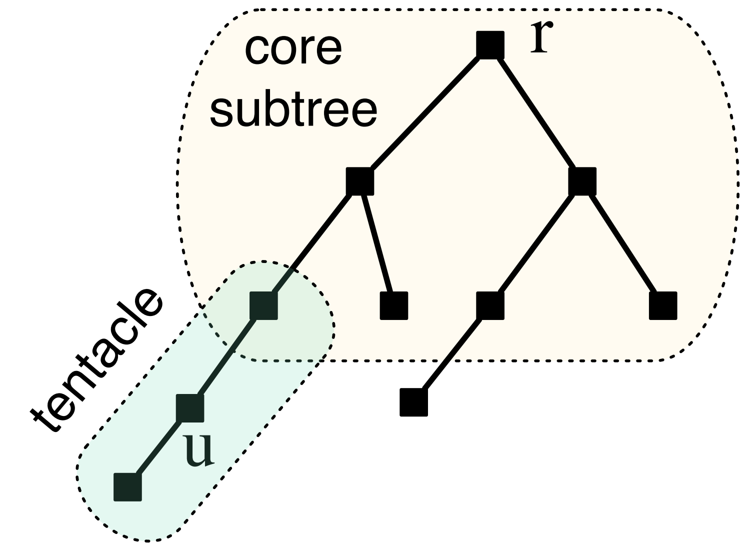


The argument

- We show that for every traversal edge collected by the strategy, the adversary releases more vertices and can increase his gain by at least 6
- If the strategy decices to stop at some point, the ratio is ≥6
- If the game lasts long enough, the adversary could stop, as his gain is quadratic in tentacle length, while strategy's gain is only linear







MinCC

Any strategy has ratio at least n-2

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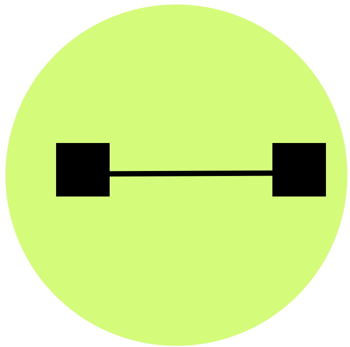

Partition produced by the algorithm

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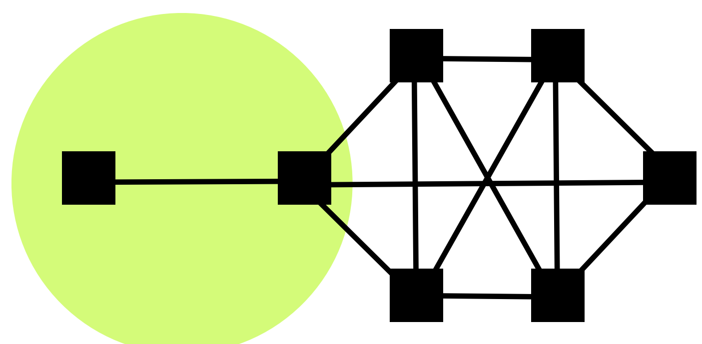
- Algorithm can merge only part of the clique. Ratio is (n-2)/1



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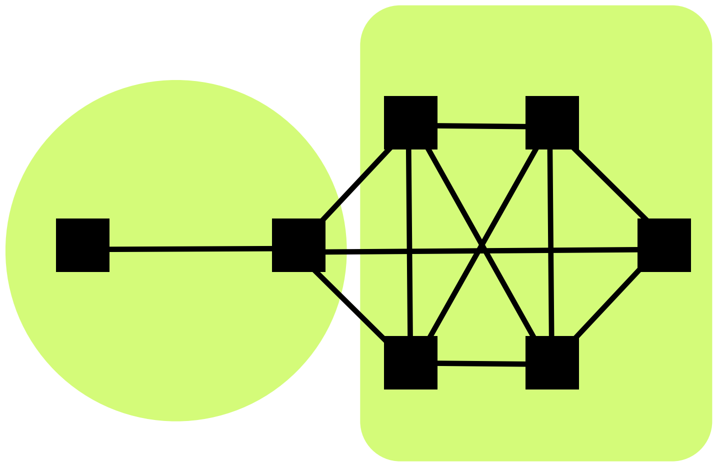

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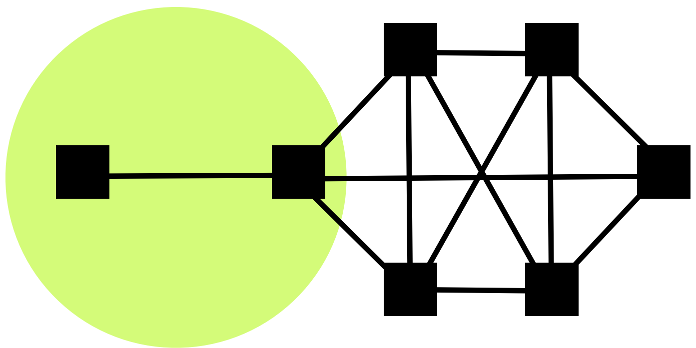

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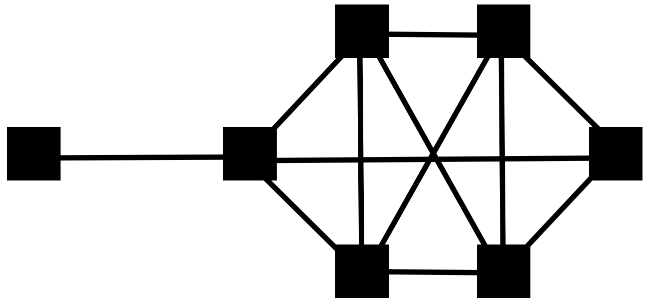

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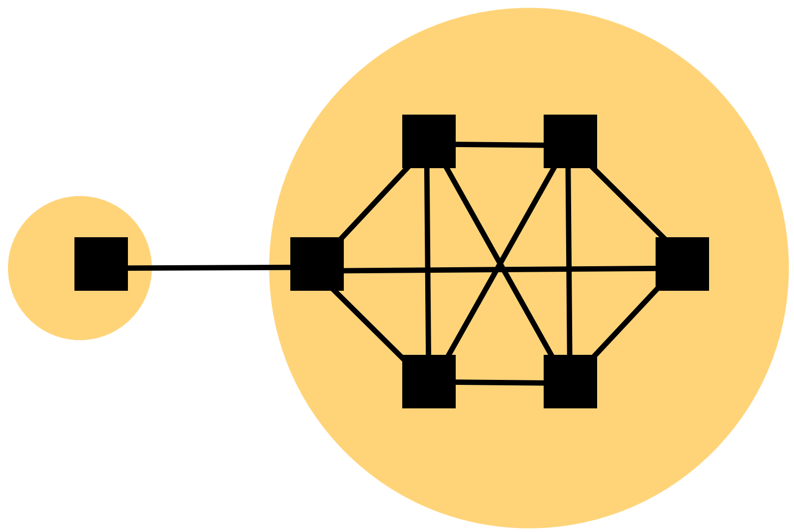
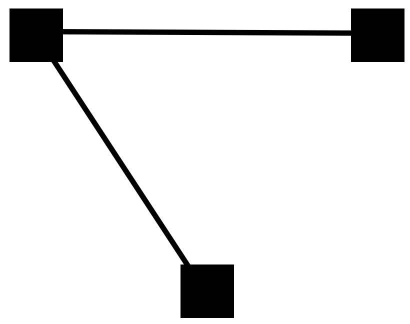


Optimal partition

Any greedy strategy has ratio at most n-2

- **greedy:** while merges are possible, the algorithm does some merge
- **key observation:** for 3 vertices connected by 2 edges, at least one of the edges is a non-clique edge



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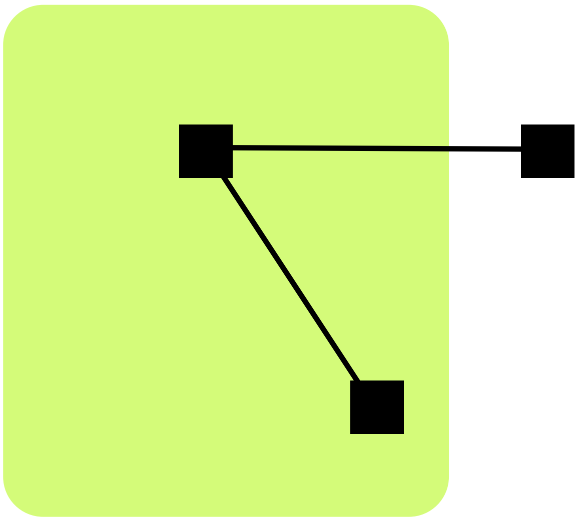

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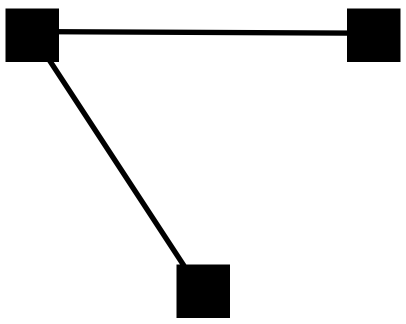
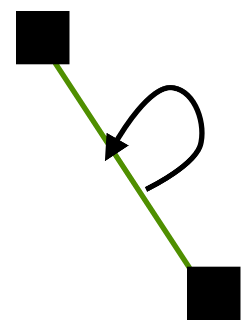
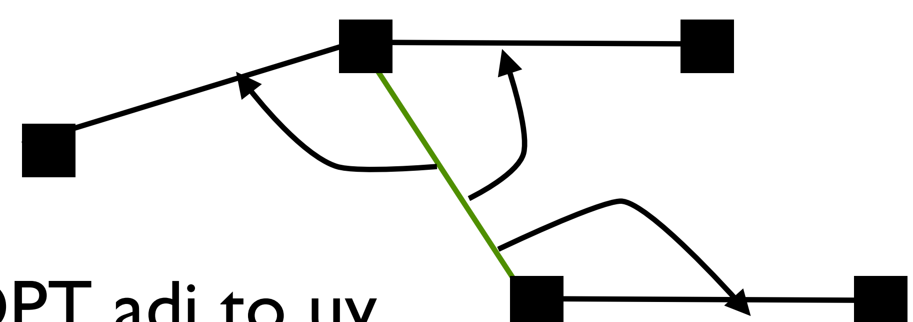


Any greedy strategy has ratio at most n-2

- **compare** a greedy solution ALG with optimal solution OPT
- **charging scheme:** charge every non-clique edge in ALG to non-clique edges by OPT
- **show:** that every non-clique edge obtains at most n-2 charges



- Given non-clique edge uv in ALG:
  - if uv is also non-clique edge in OPT: **self-charge** uv to uv
  - else **proximate charge** uv in equal fractions to all non-clique edges in OPT adj to uv



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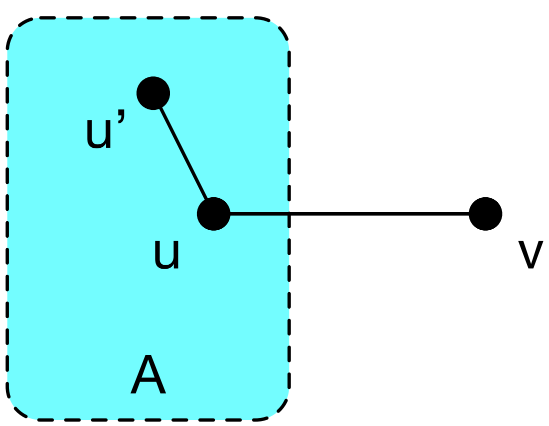
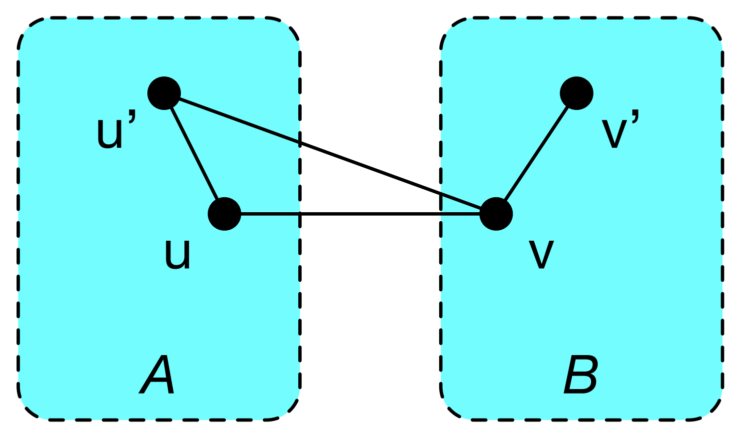
- **Claim** every non-clique edge uv in OPT has an adjacent non-clique edge in OPT
  - Say v arrived after u. Let A be a clique in ALG when v arrived with u∈A.
  - Why did Greedy not add v to A ?
  - Well the clique B containing v could not be merged with A.
  - There must be 3 vertices connected by only 2 edges:
    either uv and some vertex in A
    or v and some vertex in A and some in B





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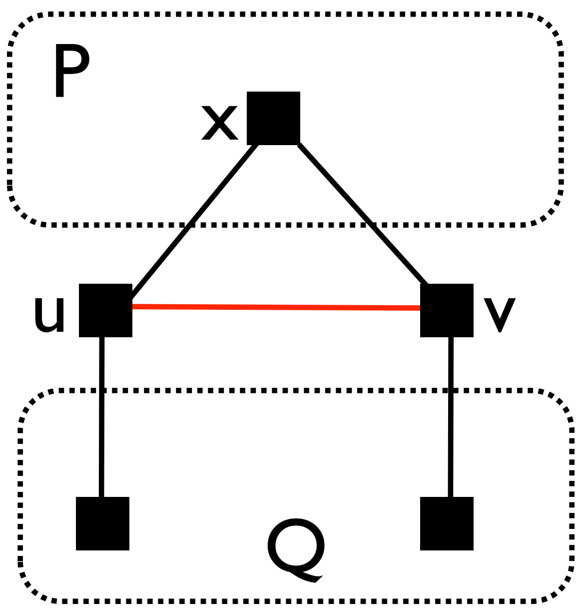
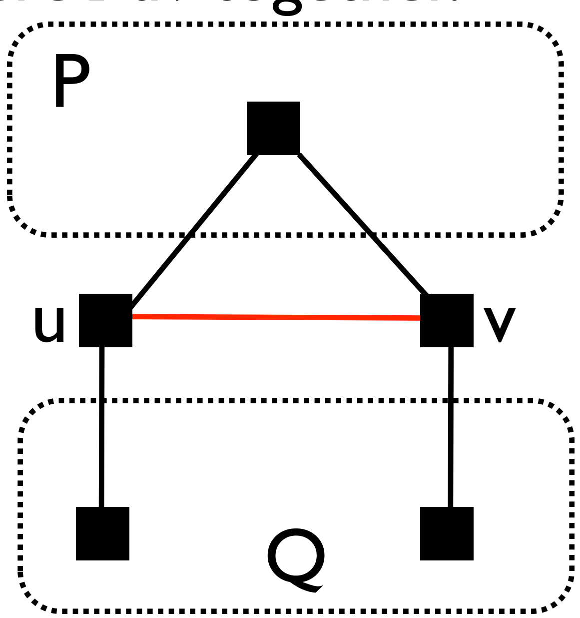
- **Count** how much charge can a non-clique edge uv in OPT obtain
  - P = set of vertices connected to both endpoints
    Q= set of vertices connected to a single endpoint
  - proximate charge from Q is at most |Q|
  - for every x∈P, one of uv or ux must be non-clique edge in OPT
    Hence if ux proximate charges to uv, then with fraction ≤ ½
  - Charge from P is at most |P|/2.



Any greedy strategy has ratio at most n-2

- **We have** $|P|+|Q|\leq n-2$
- **Claim** $|P|/2+|Q|+1 \leq n-2$
  - ok if uv is cluster edge in ALG
  - ok if $|P|\geq 2$
  - ok of Q contains cluster edges in ALG
  - case $|P|\leq 1$ and Q contains only non-cluster edges in ALG cannot happen. Greedy would have clustered uv together.



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