Introduction to online algorithms and the multi-level aggregation problem

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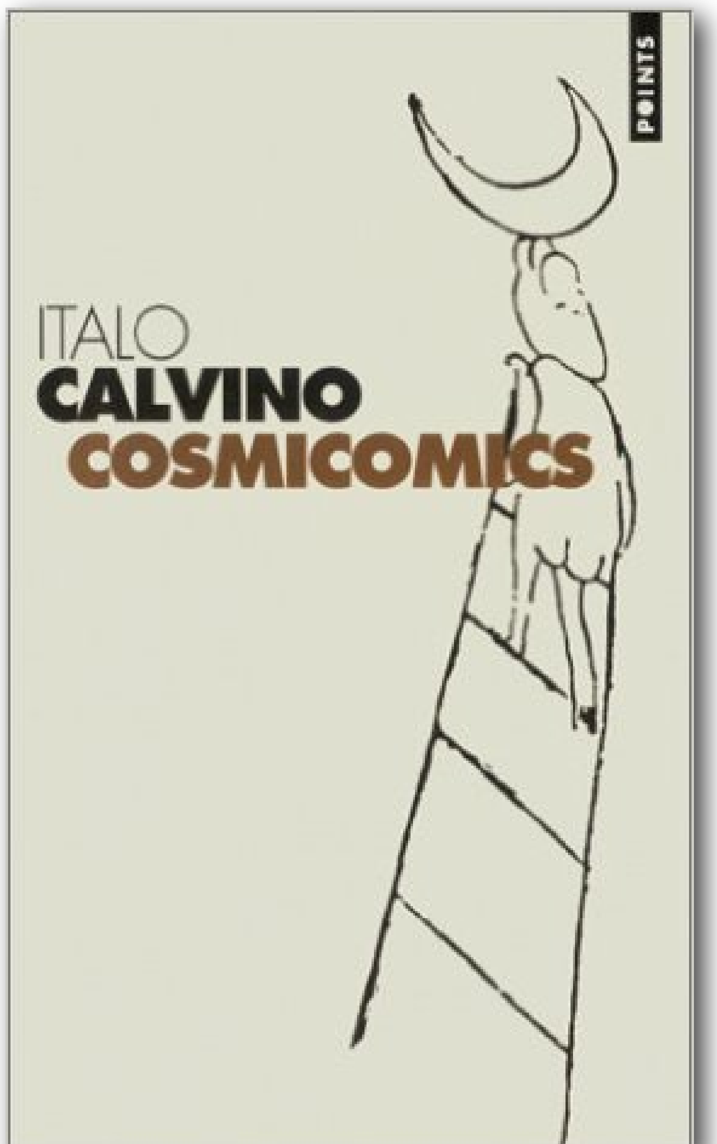
My favorite story about how the world was created



The online setting

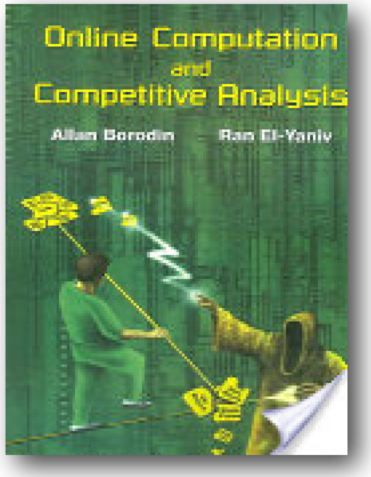
- The input is revealed to the algorithm in form of a request sequence (≠offline algorithm: input-compute-output)

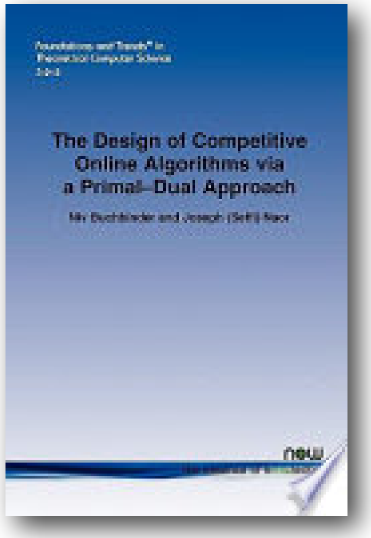
- Each request has to be served with an irrevocable decision (≠dynamic data structure)

- Some objective has to be minimized

- Performance is measured by competitive ratio.
  Algorithm A is c-competitive if for all request sequences σ
  $$A(\sigma) \leq c \cdot OPT(\sigma) + \text{constant}$$
  (=price of not knowing the future)

- Game between algorithm (make decisions to keep cost small) and adversary (generate request sequence to make cost big)



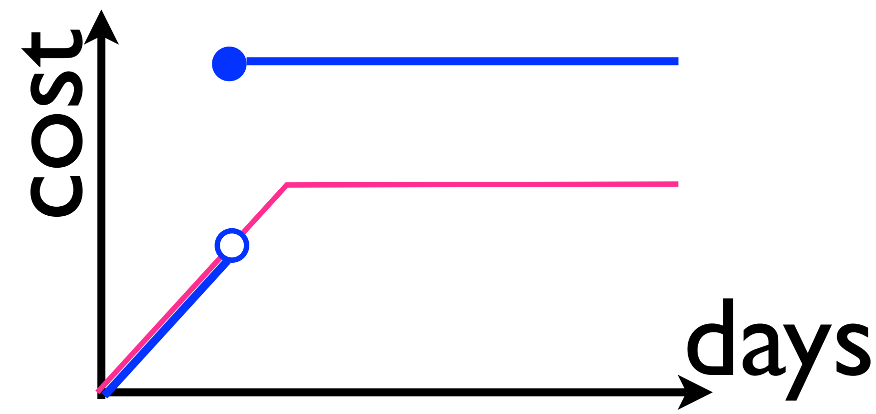


ski rental problem

- Request sequence = ski,ski,...,ski
  (length n unknown in advance)
- decision: rent (1€) or buy (b€)
- OPT = min(n,b)
- determinstic ALG = rent until day t-1, on day t buy
- worst input length is n=t, giving ratio

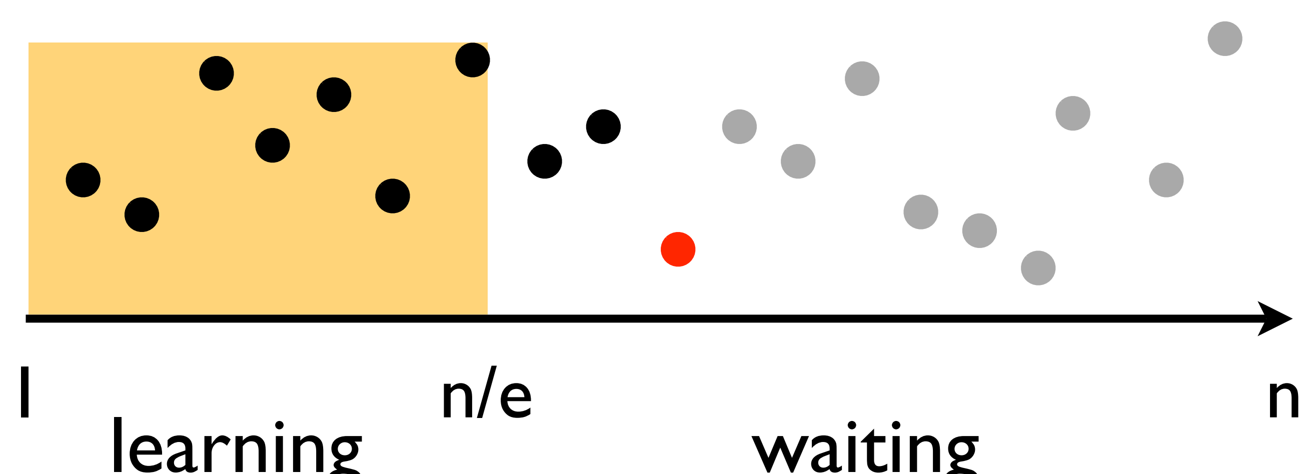
$$\frac{t-1+b}{\min(t,b)} = \frac{\min(t,b)+\max(t,b)-1}{\min(t,b)} = 1 + \frac{\max(t,b)-1}{\min(t,b)}$$

- best parameter t is t=b, giving ratio 2-1/b
- randomized algorithm achieves e/(e-1)<1.582



secretary problem: definition

- Input is a sequence of n distinct integers
  n is known,
  order is chosen uniformly at random
- On request x, algorithm can either reject or accept (and game ends)
- Goal: accept the minimum integer with high probability
- ALG: reject first n/e entries, then accept first entry that is smaller than anything seen so far
  Claim: probability of success is $1/e > 0.36$



secretary problem: analysis

- Input (ranks) is a permutation σ on 1,..,n
- set t=n/e
- Probability algorithm suceeds is

$$\sum_{j=t+1}^{n} \mathbb{P}[\sigma[j]=1 \text{ and minimum of } \sigma[1],\ldots,\sigma[j-1] \text{ is in } \sigma[1],\ldots,\sigma[t]]$$

$$= \sum_{j=t+1}^{n} \frac{1}{n} \cdot \frac{t}{j-1}$$

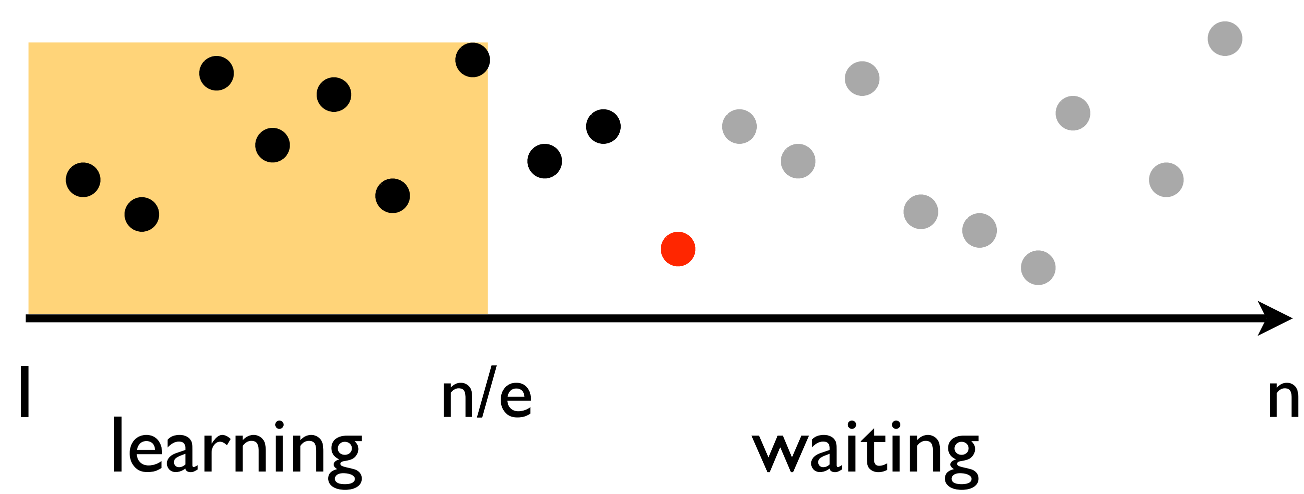
$$= \frac{t}{n} \sum_{j=t+1}^{n} \frac{1}{j-1}$$

$$= \frac{t}{n} \left(\sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{t-1} \frac{1}{j} \right)$$

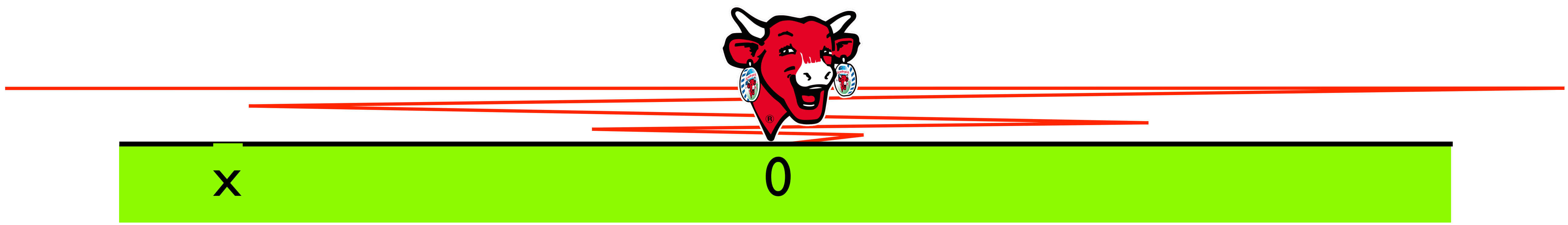
$$\sim \frac{t}{n} (\ln n - \ln t)$$

$$= \frac{t}{n} \ln \frac{n}{t}$$

- which is maximized by t=n/e evaluating to 1/e

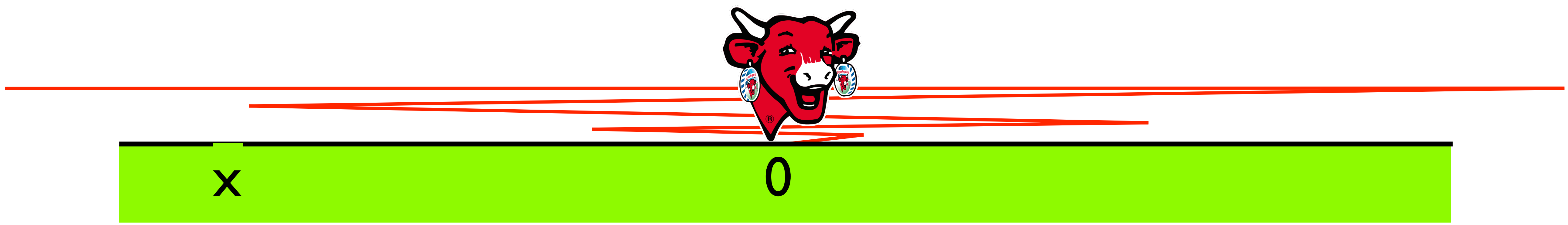


cow path problem: definition



- there is juicy grass on the other side of the fence
- cow is at position 0,
  fence has an opening at position x with $|x| \geq 1$
  (sign(x) is unknown to the cow)
- doubling ALG: walk to +1,-2,+4,-8,+16,-32,…
- competitive ratio:= distance walked to opening / $|x|$

cow path problem: analysis



- worst case: $|x|=2^i+\varepsilon$
- cost of ALG:

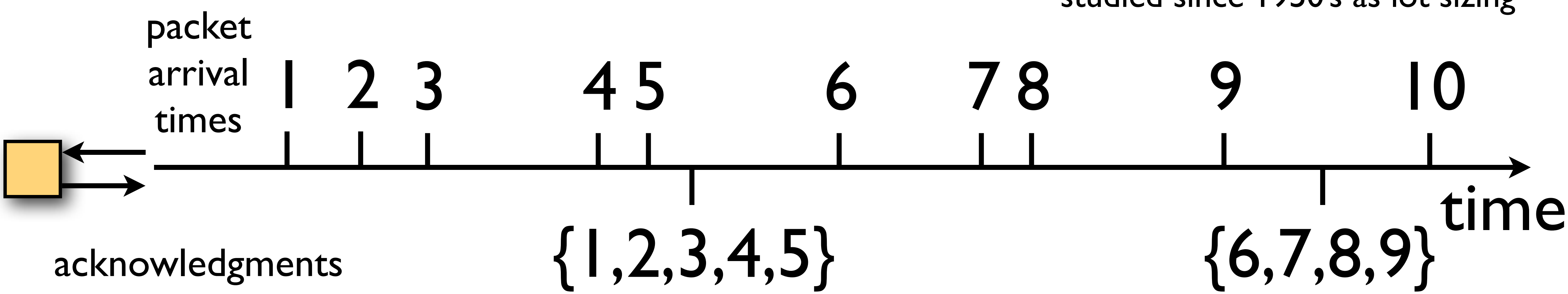
$$
\begin{aligned}
& 2(1+2+4+\ldots+2^{i+1}) + 2^i+\varepsilon \\
= \; & 2(2^{i+2}-1) + 2^i+\varepsilon \\
< \; & 8\cdot 2^i + 2^i+\varepsilon \\
< \; & 9\cdot \mathrm{OPT}
\end{aligned}
$$

Multi-Level Aggregation

Marcin Bienkowski
Martin Böhm
Jaroslaw Byrka
Marek Chrobak
C.D.
Lukáš Folwarczný
Łukasz Jeż
Jiří Sgall
Nguyễn Kim Thắng
Pavel Veselý

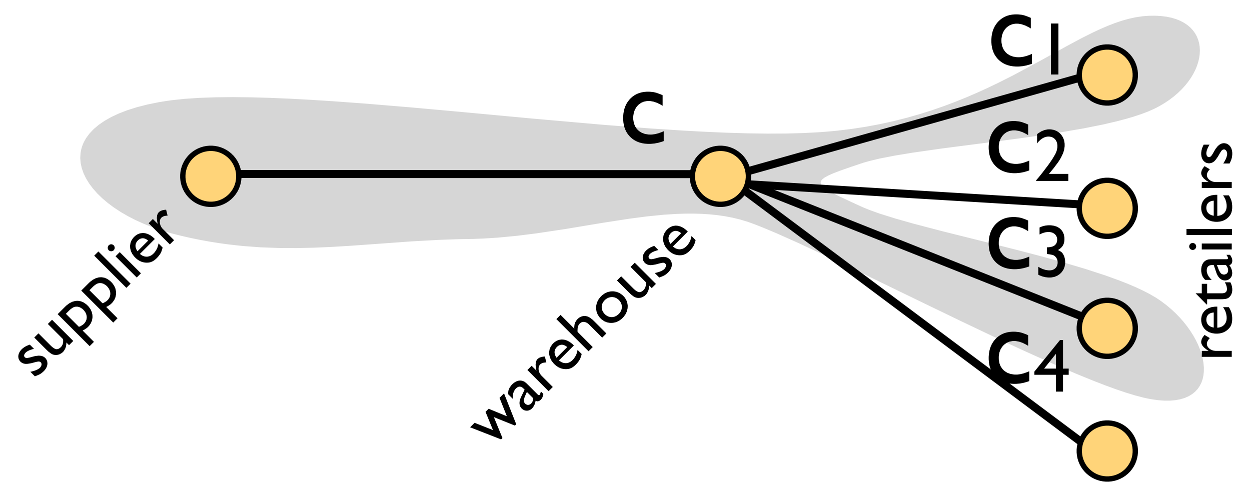
1-level aggregation: TCP acknowledgement

studied since 1950's as lot sizing



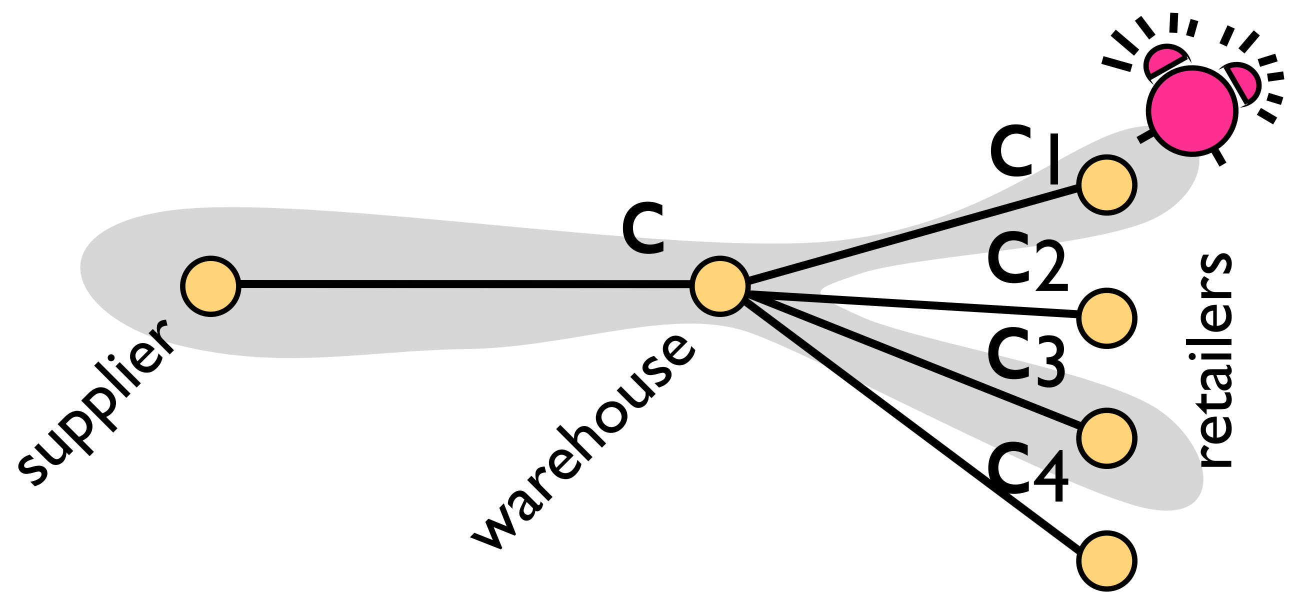
- Minimize number of acknowledgements + total waiting cost
- Offline: optimal dynamic programming solution in time O(n log n)
- Online: deterministic ratio = 2, randomized ratio = 1/e
  (similar to ski rental: send acknowledgement as soon as total waiting time reaches 1)
- The deadline variant is trivial: acknowledge as soon as a deadline of a packet expires

2-level aggregation: joint replenishment pb



- requests arrive at leafs
- general model: a request comes with an increasing waiting function of the serving time, generally just - arrival time
- deadline model: a request comes with a strict deadline and issues no waiting cost
- they are served by buying a subtree containing them
- NP-hard, even APX-hard

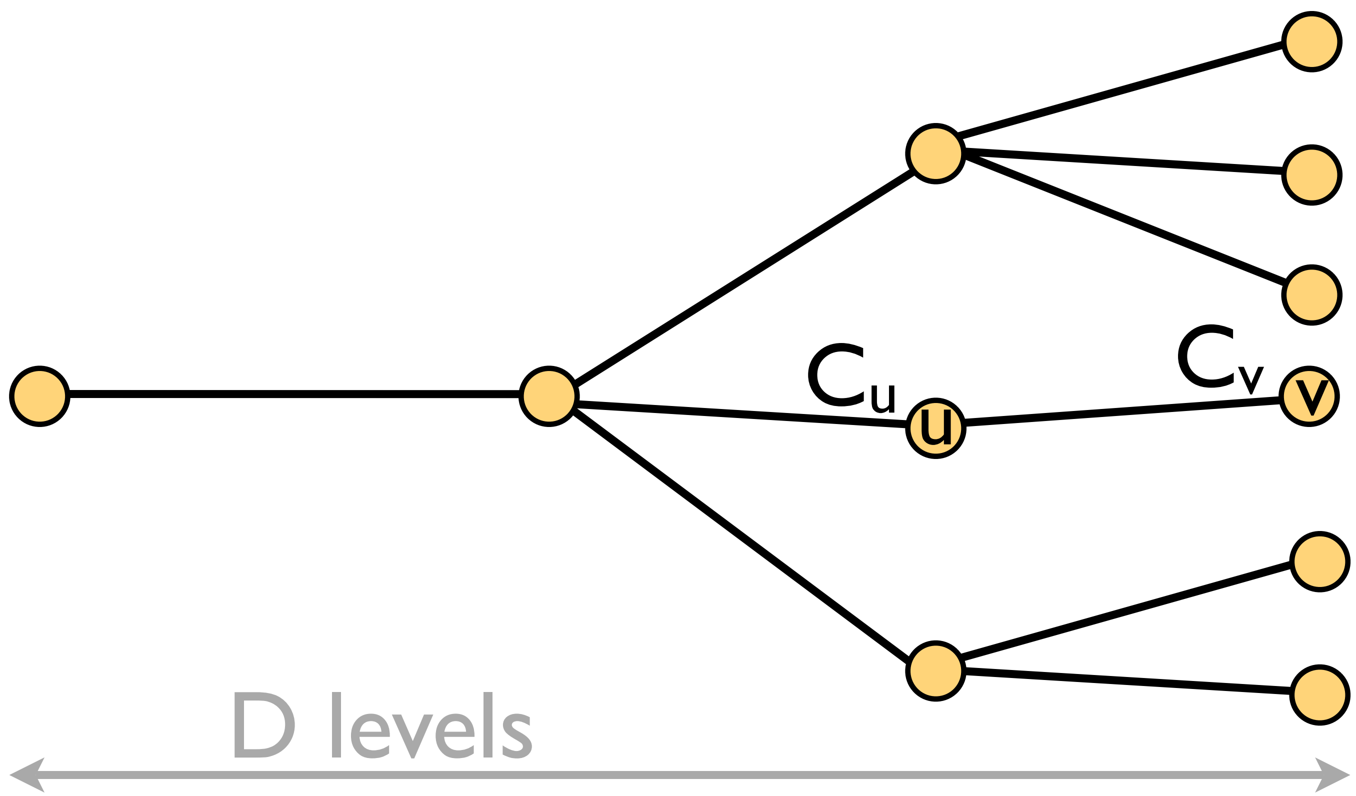
2-level aggregation:
deadline model



- A simple 2-competitive algorithm:
  As soon as some request reaches its deadline, serve at the same time a set of most urgent requests S with $\sum_{i \in S} C_i \simeq C$

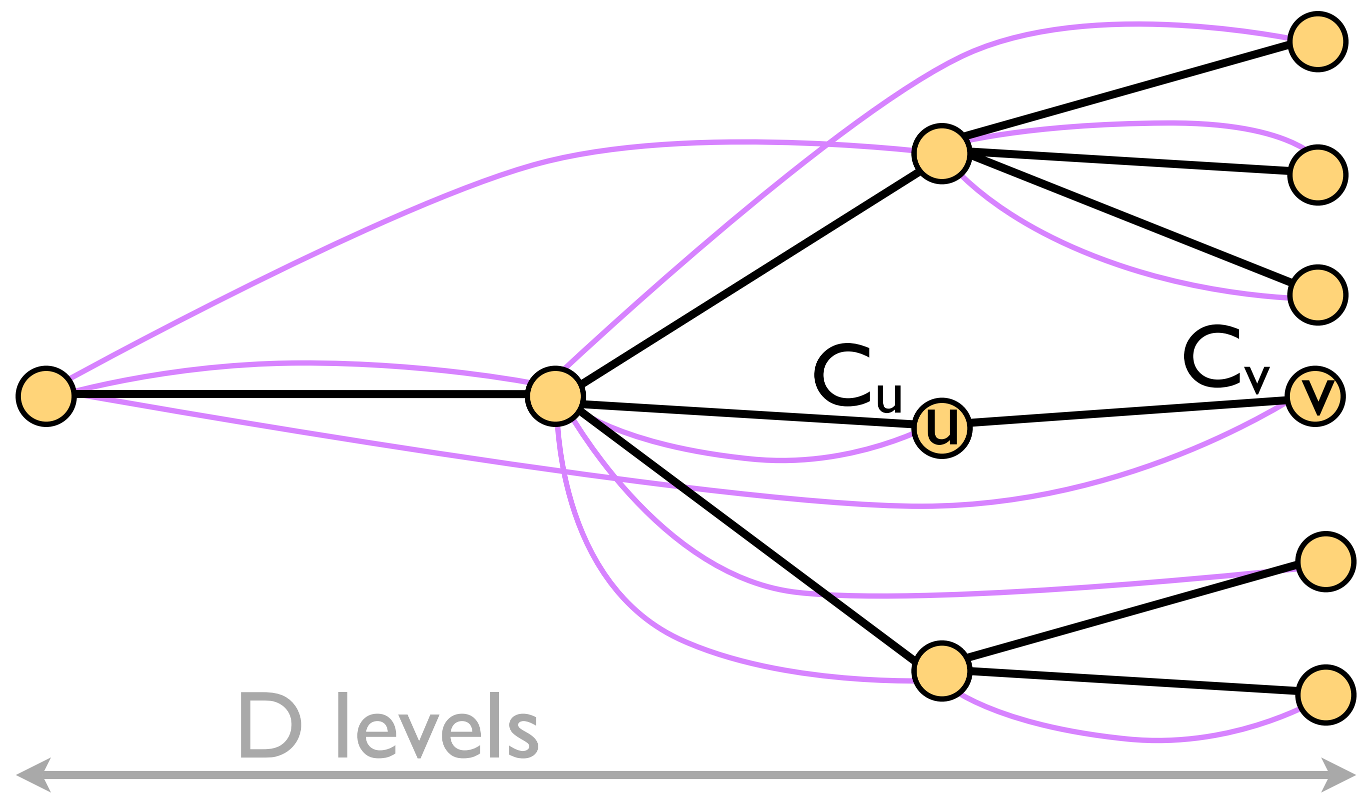
- this is optimal

L-decreasing instances



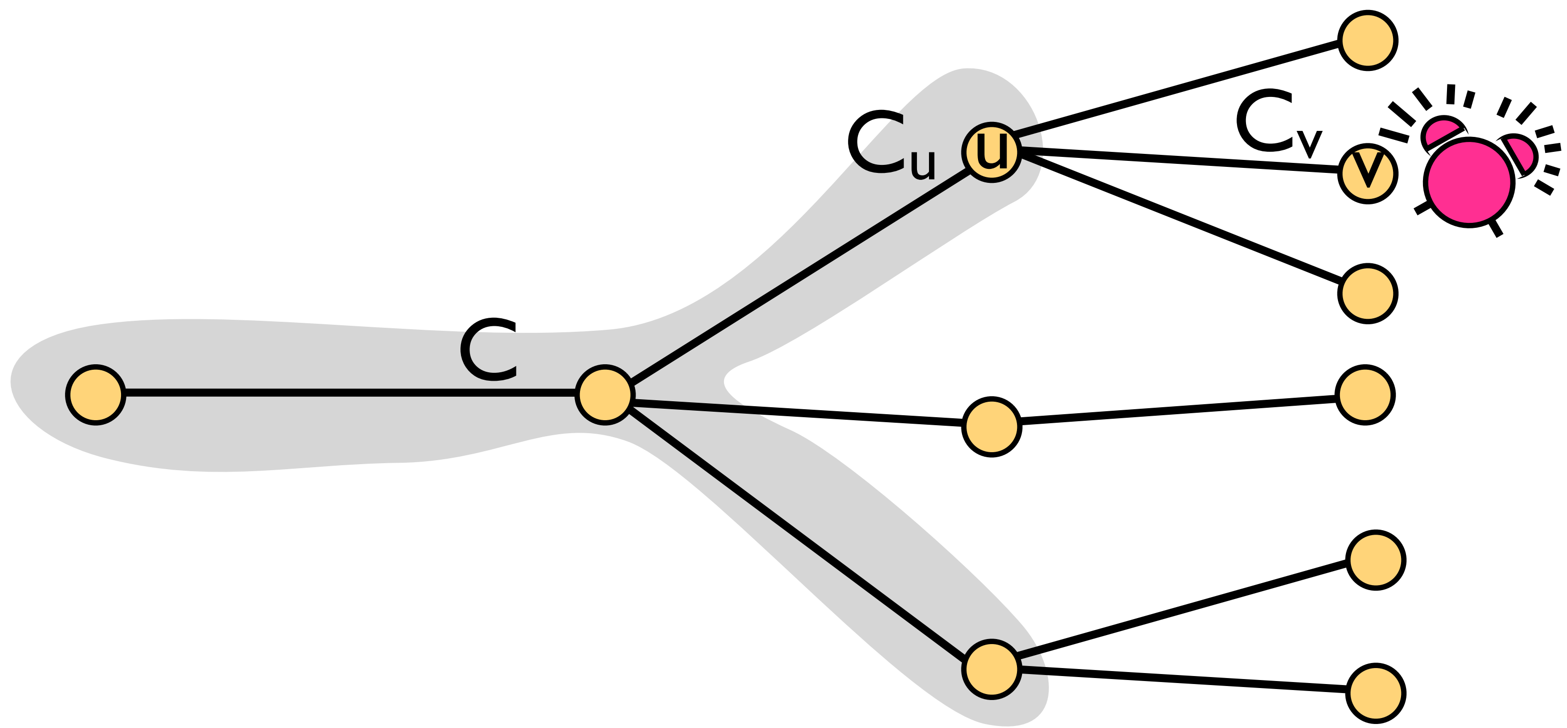
- **Definition**: instance I is L-decreasing if for every vertex v and its directed ancestor u we have $C_u \geq L \cdot C_v$
-
-

L-decreasing instances



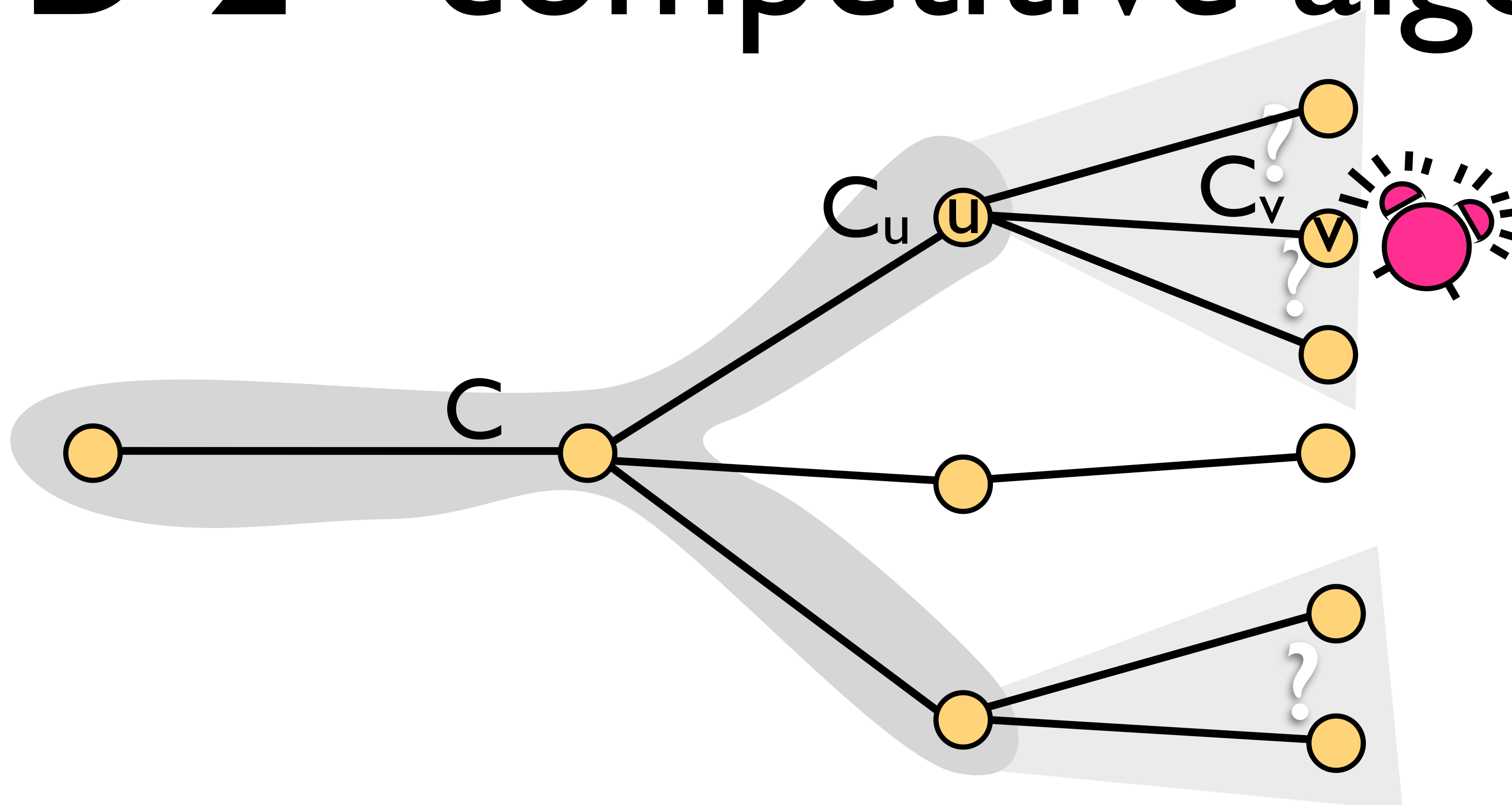
- **Definition**: instance I is L-decreasing if for every vertex v and its directed ancestor u we have $C_u \geq L \cdot C_v$
- We can construct an instance I' which is L-decreasing by replacing edges with shortcuts to closest heavy enough ancestors (or to root if none).
- **Lemma:** Every R-competitive algorithm on I' is DLR-competitive on I

Deadline model : a $D^2 2^D$-competitive algorithm



- Generalizing from 2-level trees, we add to service tree level 2 nodes of total weight $\approx$C.

Deadline model : a $D^2 2^D$-competitive algorithm

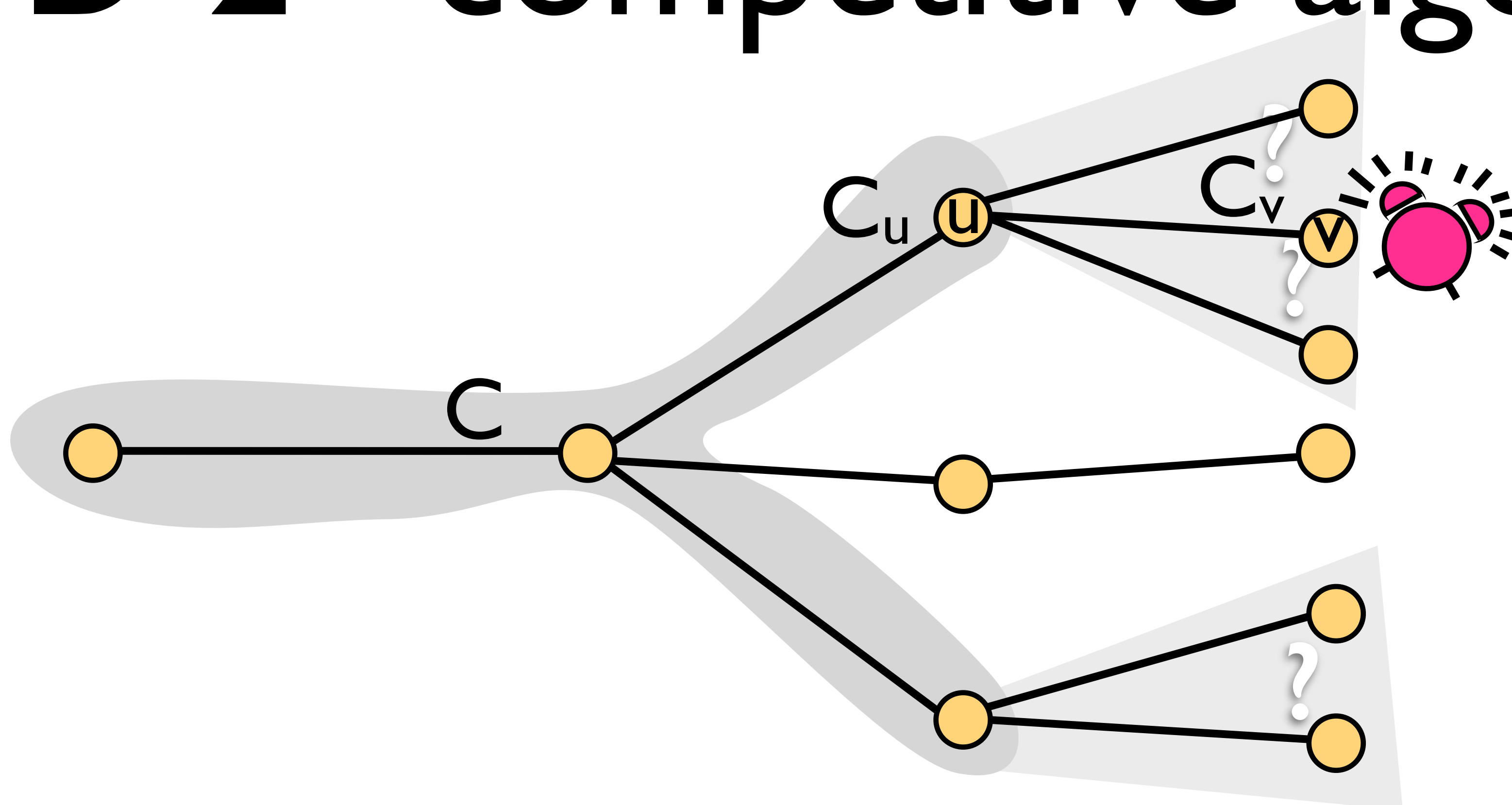


- Generalizing from 2-level trees, we add to service tree level 2 nodes of total weight ≈C. But then should we

1. add for each of these nodes u a set of most urgent leafs from subtree rooted at u of cost ≈Cu
2. or add a set of most urgent leafs from those subtrees with total weight C ?

Deadline model :

a $D^2 2^D$-competitive algorithm



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- Answer: do both.

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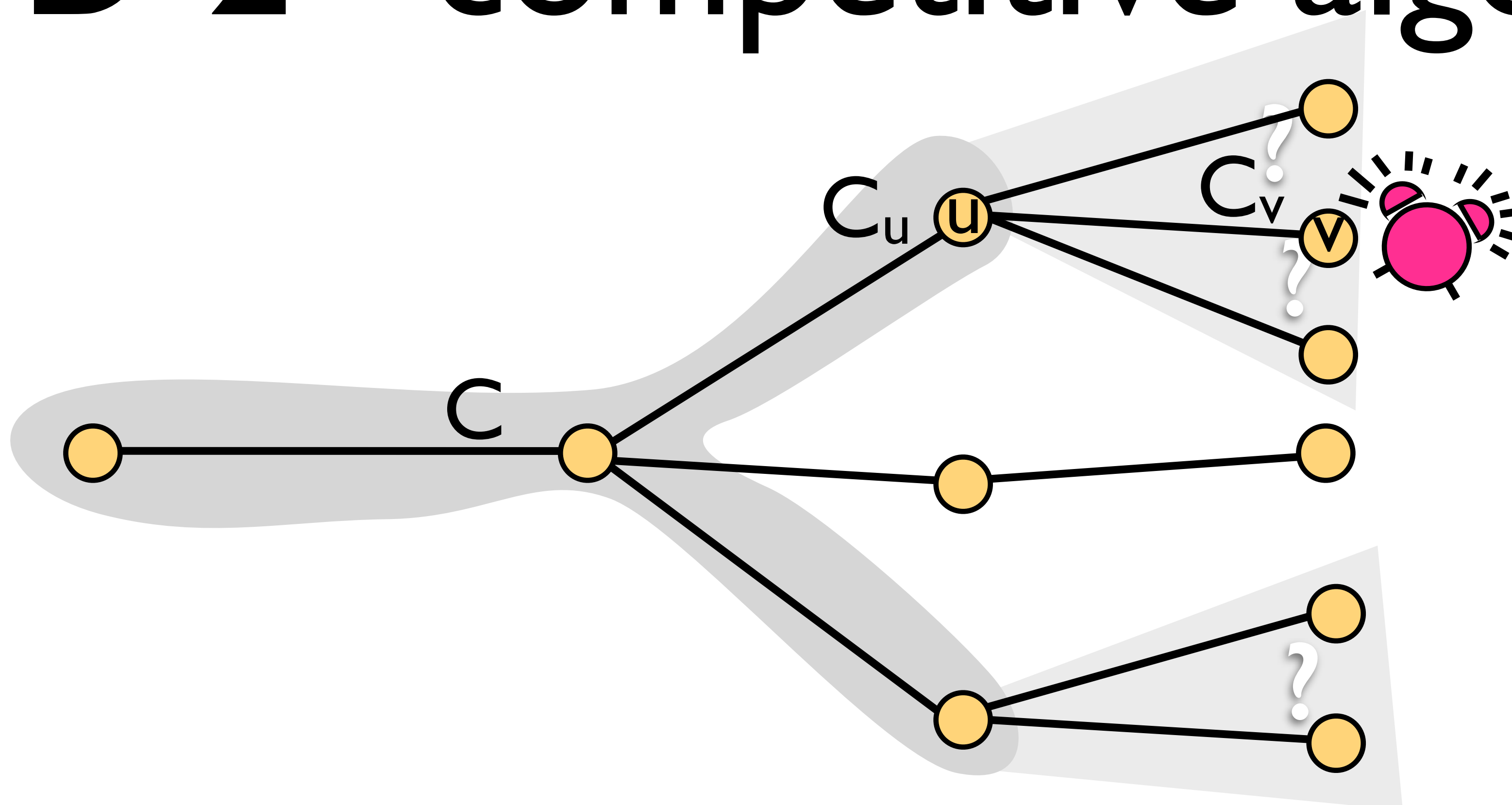

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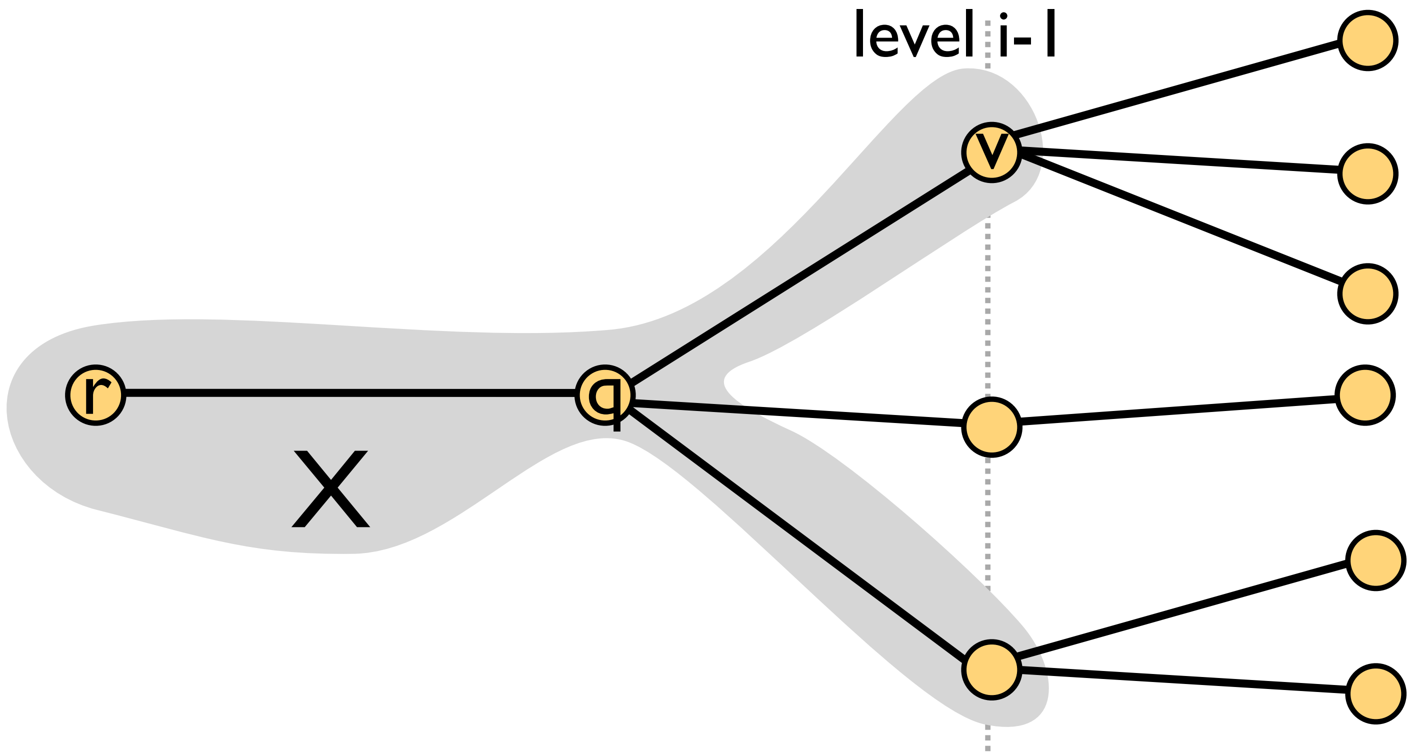
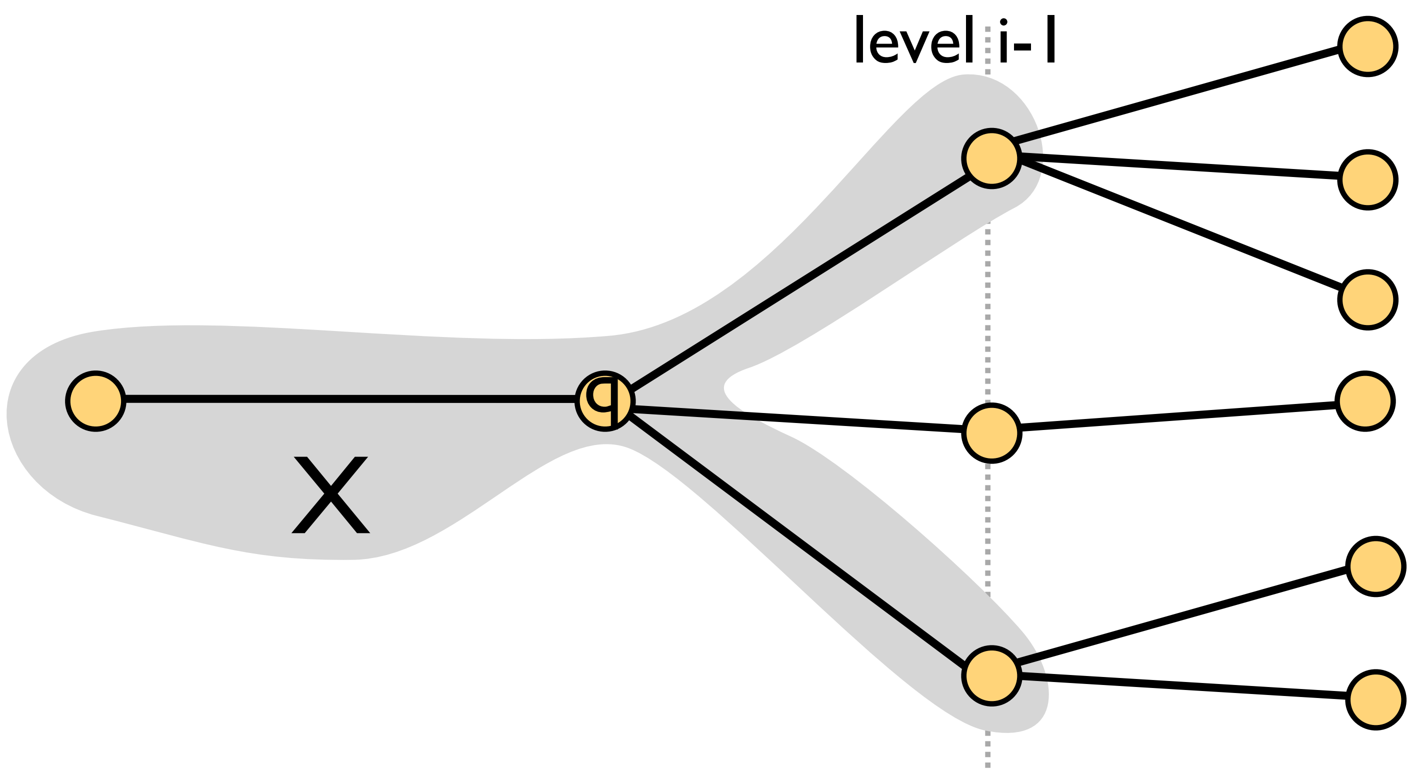


- **urgency** of a vertex u = smallest deadline among requests in subtree rooted at u

- **Urgent**(S,C) = smallest set of most urgent vertices from S with cost at least C (or S)

- ALG: **init** X = {root r, descendant of root q}
  for each level i=2,…,D:
  Z^i = all children of nodes in X^{i-1}
  for each v in $X^{<i}$:
  add Urgent(Z^i,C_v) to X

Deadline model : a $D^2 2^D$-competitive algorithm



- **Analysis:** show by induction that for i=2,…,D:
  $cost(X^{\leq i}) \leq (2+1/L)^{i-1} C_q$
- **Then** the algorithm is $(2+1/L)^{D-1}$ competitive on L-decreasing trees
- **Choosing** L=D/2, the algorithm is $D^2 2^D$ competitive on general trees

competitive ratio: our results

	linear or general waiting cost		deadline variant	
	upper	lower	upper	lower
depth 1	2	2	1	1
depth 2	3	2.754	2	2
depth $D \geq 2$	$O(D^4 2^D)$	2.754	$D^2 2^D$	2
paths of arbitrary depth	5	4	4	4

Thank you